

A CLASS OF CUBIC RAUZY FRACTALS

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ABSTRACT. In this paper, we study arithmetical and topological properties for a class of Rauzy fractals \mathcal{R}_a given by the polynomial $x^3 - ax^2 + x - 1$ where $a \geq 2$ is an integer. In particular, we prove the number of neighbors of \mathcal{R}_a in the periodic tiling is equal to 8. We also give explicitly an automaton that generates the boundary of \mathcal{R}_a . As a consequence, we prove that \mathcal{R}_2 is homeomorphic to a topological disk.

1. INTRODUCTION

In 1982, G. Rauzy [29] defined a compact subset \mathcal{E} of \mathbb{C} called classical Rauzy Fractal as

$$\mathcal{E} = \left\{ \sum_{i=0}^{+\infty} \varepsilon_i \alpha^i, \varepsilon_i \in \{0, 1\}, \varepsilon_i \varepsilon_{i+1} \varepsilon_{i+2} \neq 111, \forall i \geq 0 \right\},$$

where α is one of the two complex roots of modulus < 1 of the polynomial $P(x) = x^3 - x^2 - x - 1$.

The classical Rauzy fractal has many beautiful properties: It is a connected set, with interior simply connected, and boundary fractal. Moreover, it induces a periodic tiling of the plane \mathbb{C} modulo the group $\mathbb{Z}\alpha^{-3} + \mathbb{Z}\alpha^{-2}$.

The Rauzy fractal was studied by many mathematicians and was connected to many topics as: numeration systems ([23], [25], [27]), geometrical representation of symbolic dynamical system ([4], [5], [6], [8], [17], [22], [28], [32], [31]), multidimensional continued fractions and simultaneous approximations ([7], [10], [9], [18]), auto-similar tilings ([2], [1], [4], [27]) and Markov partitions of Hyperbolic automorphisms of Torus ([19], [22], [27]).

There are many ways of constructing Rauzy fractals, one of them is by β -expansions.

Let $\beta > 1$ be a real number and $x \in \mathbb{R}^+$. Using greedy algorithm, we can write x in base β as $x = \sum_{i=-\infty}^k a_i \beta^i$ where $k \in \mathbb{Z}$ and a_i belong to the set A where $A = \{0, \dots, \beta - 1\}$ if $\beta \in \mathbb{N}$ or $A = \{0, \dots, \lfloor \beta \rfloor\}$ otherwise, where $\lfloor \beta \rfloor$ is the integer part of β . The sequence $(a_i)_{i \leq k}$ is called β -expansion of x and is also denoted by $a_k a_{k-1} \dots$. The greedy algorithm can be defined as follows (see [26] and [14]): denote by $\{y\}$ the fractional party of a number y . There exists an integer $k \in \mathbb{Z}$ such $\beta^k \leq x < \beta^{k+1}$. Let $x_k = \lfloor x/\beta^k \rfloor$ and $r_k = \{x/\beta^k\}$. Then for $i < k$, put $x_i = \lfloor \beta r_{i+1} \rfloor$ and $r_i = \{\beta r_{i+1}\}$. We get

$$x = x_k \beta^k + x_{k-1} \beta^{k-1} + \dots$$

if $k < 0$ ($x < 1$), we put $x_0 = x_{-1} = \dots = x_{k+1} = 0$. If an expansion $(x_i)_{i \leq k}$ satisfies $x_i = 0$ for all $i < n$, it is said to be finite and the ending zeros are omitted. It will be denoted by $(x_i)_{n \leq i \leq k}$ or $x_k \dots x_n$.

Now, assume that β is a Pisot number of degree $d \geq 3$, that means that β is an algebraic integer of degree d whose Galois' conjugates have modulus less than one. We denote by b_2, \dots, b_r the real Galois conjugates of β and by $\beta_{r+1}, \dots, \beta_{r+s}, \beta_{r+s+1} = \bar{\beta}_{r+1}, \dots, \beta_{r+2s} = \bar{\beta}_{r+s}$ its complex Galois conjugates. Let $\psi = (\beta_2, \dots, \beta_{r+s}) \in \mathbb{R}^{r-1} \times \mathbb{C}^s$ and put $\psi^i = (\beta_2^i, \dots, \beta_{r+s}^i)$ for all $i \in \mathbb{Z}$.

The Rauzy fractal is by definition the set

$$\mathcal{R} = \mathcal{R}_\beta = \left\{ \sum_{i=0}^{+\infty} a_i \psi^i, (a_i)_{i \geq 0} \in E_\beta \right\},$$

where

$$E_\beta = \{(x_i)_{i \geq k}, k \in \mathbb{Z} \mid \forall n \geq k, (x_i)_{n \geq i \geq k} \text{ is a finite } \beta \text{ expansion}\}.$$

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Observe that \mathcal{R}_β is a compact subset of $\mathbb{R}^{r-1} \times \mathbb{C}^s \approx \mathbb{R}^{d-1}$.

For example, if $\beta > 1$ is a root of the polynomial $P(x) = x^3 - x^2 - x - 1$, we obtain the classical Rauzy fractal $\mathcal{R}_\beta = \mathcal{E}$.

An important class of Pisot numbers are those such that the associated Rauzy fractal has 0 as an interior point. These numbers were characterized by Akiyama in [3]. They are exactly the Pisot numbers that satisfy

$$\mathbb{Z}[\beta] \cap [0, +\infty[\subset \text{Fin}(\beta) \text{ (called property (F)) ,}$$

where $\text{Fin}(\beta)$ is the set of nonnegative real numbers which have a finite β -expansion.

In this paper we study properties of the Rauzy fractal associated to a class of cubic unit Pisot numbers that satisfy property (F). These numbers were characterized in [1] as being exactly the set of dominant roots of the polynomial (with integers coefficients)

$$P_{a,b}(x) = x^3 - ax^2 - bx - 1, \quad a \geq 0, \quad -1 \leq b \leq a + 1.$$

(If $b = -1$ add the restriction $a \geq 2$).

In particular, this set divided into three subsets:

- a) $0 \geq b \geq a$, and in this case $d(1, \beta) = \cdot ab1$.
- b) $b = -1, a \geq 2$. In this case $d(1, \beta) = \cdot (a-1)(a-1)01$.
- c) $b = a+1$, and in this case $d(1, \beta) = \cdot (a+1)00a1$, where $d(1, \beta)$ is the Rényi β -representation of 1 (see [30]).

Geometrical and arithmetical properties of the Rauzy fractal associated to polynomials $P_{a,b}$, $a \geq b \geq 1$ were studied in [20]. Here we will study the case $a \geq 2, b = -1$. In this case the polynomial $p(x) = x^3 - ax^2 + x - 1 = (x - \beta)(x - \alpha)(x - \gamma)$, where $\beta > 1$ and $\alpha, \gamma \in \mathbb{C} \setminus \mathbb{R}$, and the Rauzy fractal

$$\mathcal{R}_a = \left\{ \sum_{i=0}^{\infty} a_i \alpha^i, \quad a_i a_{i-1} a_{i-2} a_{i-3} <_{lex} d(1, \beta) = (a-1)(a-1)01, \quad \forall i \geq 0 \text{ where } a_{-1} = a_{-2} = a_{-3} = 0 \right\},$$

where $<_{lex}$ is the lexicographic order on finite words.

On the other hand, consider the sequence $R_0 = 1, R_1 = a, R_2 = a^2, R_{n+3} = aR_{n+2} - R_{n+1} + R_n \quad \forall n \geq 0$. It is known, using greedy algorithm that for all nonnegative integer n can be written as $n = \sum_{i=0}^N a_i R_i$. The sequence $(a_i)_{0 \leq i \leq N}$ is called a greedy R-expansion.

The Rauzy fractal is equal

$$\mathcal{R}_a = \left\{ \sum_{i=0}^{\infty} a_i \alpha^i, \quad \forall N \geq 0 \quad (a_i)_{0 \leq i \leq N} \text{ is a greedy R-expansion} \right\}.$$

We will also study properties of another set very closed to the Rauzy Fractal. We call this set the G -Rauzy fractal and define it by

$$\mathcal{G}_a = \left\{ \sum_{i=0}^{\infty} a_i \alpha^i, \quad \forall N \geq 0, \quad (a_i)_{0 \leq i \leq N} \text{ is a greedy } G\text{-expansion} \right\},$$

where $G = (G_n)_{n \geq 0}$ where $G_0 = 1, G_1 = a, G_2 = a^2 + b, G_{n+3} = aG_{n+2} + bG_{n+1} + G_n \quad \forall n \geq 0$.

The set \mathcal{G} was defined in [18] by Hubert and Messaoudi. They used it to prove that $(G_n)_{n \geq 0}$ is the sequence of best approximations of the vector $(1/\beta, 1/\beta^2)$ (for a certain norm on \mathbb{R}^2 called the Rauzy norm \mathcal{N}).

In the case where $b = -1$ and $a \geq 2$ it is known (see [18]) that the set of G -expansions is equal to the set of $(\varepsilon_i)_{0 \leq i \leq N}$ that satisfy the following conditions:

$$\varepsilon_i \varepsilon_{i-1} \varepsilon_{i-2} \varepsilon_{i-3} <_{lex} d(1, \beta) = (a-1)(a-1)01, \quad \forall i \geq 3,$$

and the initial conditions

$$\varepsilon_0 < a, \quad \varepsilon_1 \varepsilon_0 <_{lex} (a-1)(a-1), \quad \varepsilon_2 \varepsilon_1 \varepsilon_0 <_{lex} (a-1)(a-1)0.$$

Observe that the above initial conditions from the fact that: $\varepsilon_0 G_0 < G_1 = a, \varepsilon_0 G_0 + \varepsilon_1 G_1 <_{lex} G_2 = (a-1)G_1 + (a-1)G_0, \varepsilon_0 G_0 + \varepsilon_1 G_1 + \varepsilon_2 G_2 < G_3 = (a-1)G_2 + (a-1)G_1$.

Many topological properties of \mathcal{R}_a are known (see [1, 16, 22, 25, 29]): It's a connected compact subset of \mathbb{C} , with interior simply connected and fractal boundary, moreover it induces a periodic tiling of the plane modulo \mathbb{C} . It can be also seen as geometrical realization of the dynamical system associated to the substitution σ defined by: $\sigma(1) = 1^{a-1}2, \sigma(2) = 1^{a-1}3, \sigma(3) = 4, \sigma(4) = 1$.

To our knowledge, geometrical and topological properties of the set \mathcal{G}_a were not yet studied. In this paper, we show that \mathcal{G}_a induces a periodic tiling of the complex plane. We also construct an explicit finite state automaton \mathcal{A} that generates both boundaries of \mathcal{R}_a and \mathcal{G}_a . With this we prove that for all $a \geq 2$, \mathcal{R}_a has 8 neighbors while \mathcal{G}_a has 6 neighbors (in the periodic tiling). The interest of giving explicitly the automaton \mathcal{A} remains in the fact that the study of properties of \mathcal{A} give topological and metrical information about the boundary \mathcal{G}_a and \mathcal{R}_a .

Here, we prove that the boundary of \mathcal{R}_2 is homeomorphic to a topological circle. This study can be done for all integer $a \geq 2$.

The paper is divided by the following manner. In the second section, we give some notations. In the third section, we study some properties of the boundary of \mathcal{G}_a , in the fourth section, we construct an explicit finite state automaton that recognizes the boundaries of \mathcal{G}_a and \mathcal{R}_a for all $a \geq 2$. The fifth section is devoted to the study topological properties of the boundary of \mathcal{R}_2 . In particular, using the automaton, we prove that the boundary of \mathcal{R}_2 is homeomorphic to a circle.

2. NOTATIONS AND DEFINITIONS

Denote by $E(G)$ (resp. $E(R)$) the set of sequences $(a_n)_{n \in \mathbb{Z}}$ belonging to $\{0, 1, \dots, a-1\}^{\mathbb{Z}}$ such that, there exists an integer $k \in \mathbb{Z}$ satisfying $a_k > 0$ and $a_n = 0$ for all $n < k$, moreover for all $p \geq k$, the sequence $(a_n)_{k \leq n \leq p}$ is a G -expansion (resp. R -expansion). That is

$E(R) = \{(a_n)_{n \in \mathbb{Z}}, \exists k \in \mathbb{Z}, a_k > 0, a_i = 0 \text{ for all } i < k, a_i a_{i-1} a_{i-2} a_{i-3} <_{lex} (a-1)(a-1)01, \forall i \geq k\}$, and $E(G) = \{(a_n)_{n \in \mathbb{Z}}, \exists k \in \mathbb{Z}, a_k > 0, a_i = 0 \text{ for all } i < k, a_i a_{i-1} a_{i-2} a_{i-3} <_{lex} (a-1)(a-1)01, \forall i \geq k, a_k < a, a_{k+1} a_k <_{lex} (a-1)(a-1), a_{k+2} a_{k+1} a_k <_{lex} (a-1)(a-1)0\}$. Observe that $E(G) \subset E(R)$.

We will identify a sequence $(a_n)_{n \in \mathbb{Z}}$ belonging to $E(R)$ such that $a_n = 0$ for all $n < k$ with the sequence $(a_n)_{n \geq k}$.

Let $(a_n)_{n \geq k}$ be an element of $E(R)$. Assume that there exists $p \in \mathbb{Z}$ such that for all $n > p$, $a_n = 0$. This sequence will be denoted by $(a_n)_{k \leq n \leq p}$.

For technical reasons, we will consider

$$\mathcal{R}_a = \left\{ \sum_{i=2}^{\infty} a_i \alpha^i, a_i a_{i-1} a_{i-2} a_{i-3} <_{lex} (a-1)(a-1)01, \forall i \geq 2, \text{ where } a_1 = a_0 = a_{-1} = 0 \right\}$$

and

$$\mathcal{G}_a = \left\{ \sum_{i=2}^{\infty} a_i \alpha^i, \forall N \geq 2, (a_i)_{2 \leq i \leq N} \text{ is a Greedy } G\text{-expansion} \right\}.$$

3. PROPERTIES OF \mathcal{G}_a AND ITS BOUNDARY

Theorem 3.1. *The set \mathcal{G}_a induces a periodic tiling of the complex plane, that is,*

- a) $\mathbb{C} = \bigcup_{u \in \mathbb{Z} + \mathbb{Z}\alpha} (\mathcal{G}_a + u)$;
- b) $\text{int}(\mathcal{G}_a + u) \cap (\mathcal{G}_a + v) \neq \emptyset$, $u, v \in \mathbb{Z} + \mathbb{Z}\alpha$ implies que $u = v$.

Remark 3.2. *The proof can be deduced from [29] (done in case of Rauzy fractal \mathcal{E} , see also [11]). For clarity, we will give the proof here.*

□

Consider the sequence $G' = (G'_n)_{n \geq 0}$ by $G'_0 = 0$, $G'_1 = 0$, $G'_2 = 1$, $G'_{n+3} = aG'_{n+2} - G'_{n+1} + G'_n, \forall n \geq 0$. Then $G_n = G'_{n+2}$ for all integer $n \in \mathbb{N}$.

Proposition 3.3. *The following properties are valid:*

- i) All natural integer n can be written by unique way as $n = \sum_{i=2}^N \varepsilon_i G'_i$ where $(\varepsilon_i)_{2 \leq i \leq N} \in E(G)$.
- ii) Let $(a_i)_{l \leq i \leq N}$ and $(b_i)_{l' \leq i \leq \infty}$ be two elements of $E(R)$ (resp. $E(G)$) such that $a_l > 0$ and $b_{l'} > 0$. If $\sum_{i=l}^N a_i \alpha^i = \sum_{i=l'}^{\infty} b_i \alpha^i$ then $l = l'$ and for all $i \geq N$, $b_i = 0$ and for all $l \leq i \leq N$, $a_i = b_i$.
- iii) Let $(\varepsilon_i)_{2 \leq i \leq N} \in E(R)$ (resp. $E(G)$) then $\sum_{i=2}^N \varepsilon_i \alpha^i \in \text{int}(\mathcal{R})$ (resp. $\text{int}(\mathcal{G})$). In particular, $0 \in \text{int}(\mathcal{R})$ (resp. $0 \in \text{int}(\mathcal{G})$).
- iv) Let $z \in \mathbb{Z}[\beta] \cap \mathbb{R}^+$ then there exist a sequence $(a_i)_{k \leq i \leq l} \in E(R)$, $k \leq l$ such that $z = \sum_{i=k}^l a_i \beta^i$.

- v) For all $n \geq 2$ we have $\beta^n = G'_n \beta^2 + (G'_{n-2} - G'_{n-1})\beta + G'_{n-1}$. In particular if $(\varepsilon_i)_{2 \leq i \leq l} \in E(G)$ then $\sum_{i=2}^l \varepsilon_i \beta^i = n\beta^2 + r(n)\beta + s(n)$ where $n = \sum_{i=2}^l \varepsilon_i G'_i$, $r(n) = \sum_{i=2}^l \varepsilon_i (G'_{i-2} - G'_{i-1})$ and $s(n) = \sum_{i=2}^l \varepsilon_i G'_{i-1}$.
- vi) Let $(a_i)_{l \leq i \leq k}$ and $(b_i)_{l \leq i \leq k}$ be elements of $E(G)$ (resp. $E(R)$). Then $\sum_{i=l}^k a_i \beta^i < \sum_{i=l}^k b_i \beta^i$ if, only if $(a_i)_{l \leq i \leq k} <_{lex} (b_i)_{l \leq i \leq k}$.
- vii) Let $c, d \in \mathbb{R}$ such that $\alpha^2 = c + d\alpha$, then 1, c and d are \mathbb{Q} -Linearly independent.

Remark 3.4. The results given in Proposition 3.3 are classical. For i) and vi), see [21]. For ii) see [18]. The results iii) and vii) can be found in [1].

For iv), see [13]. v) is left to the reader and can be done by induction.

Proof of Theorem 2.1.

Let $z \in \mathbb{C}$ and $\epsilon > 0$. Using item (vii) of Proposition 3.3 and Kronecker's Theorem, we deduce that the set $\{n\alpha^2 + p\alpha + q, n \in \mathbb{N}, p, q \in \mathbb{Z}\}$ is dense in \mathbb{C} . Then there exists a sequence $(z_k)_{k \geq 0} \in \mathbb{C}$ such that

$$z_k = n_k \alpha^2 + p_k \alpha + r_k, \quad n_k \in \mathbb{N}, \quad p_k, q_k \in \mathbb{Z}$$

and for all $k \geq k_0$, $|z_k - z| < \epsilon$. Let $A_k = n_k \alpha^2 + r(n_k)\alpha + s(n_k)$, where $r(n_k)$ and $s(n_k)$ are defined in item (v) in Proposition 3.3. We have $A_k \in \mathcal{G}_a$.

On the other hand,

$$A_k = n_k \alpha^2 + p_k \alpha + r_k + (r(n_k) - p_k)\alpha + (s(n_k) - r_k) = z_k + t_k \alpha + m_k,$$

where $t_k = r(n_k) - p_k$ e $m_k = s(n_k) - r_k$. Then, $z_k + t_k \alpha + m_k \in \mathcal{G}_a$.

On the other hand, for all $k \geq k_0$,

$$|z + t_k \alpha + m_k| \leq |z - z_k| + |z_k + t_k \alpha + m_k| < \epsilon + d,$$

where d is the diameter of \mathcal{G}_a . Since $\mathbb{Z} + \mathbb{Z}\alpha$ is a lattice, there exists an increasing sequence $(k_i)_{i \geq 1}$ of integer numbers such that for all $i, j \in \mathbb{N}$, $t_{k_i} \alpha + r_{k_i} = t_{k_j} \alpha + r_{k_j}$. Then, there exist $t, r \in \mathbb{Z}$ such that $t_{k_i} = t_{k_j} = t$ and $r_{k_i} = r_{k_j} = r$, for all $i, j \in \mathbb{N}$. As $z_{k_i} + t_{k_i} \alpha + m_{k_i} \in \mathcal{G}_a$, $\lim_{i \rightarrow +\infty} z_{k_i} = z$ and \mathcal{G}_a is a closed set, we have that $z + t\alpha + r \in \mathcal{G}_a$. \square

To prove, item b), it is sufficient to establish that if $(\text{int}(\mathcal{G}_a) + u) \cap \mathcal{G}_a \neq \emptyset$ where $u \in \mathbb{Z} + \mathbb{Z}\alpha$ then $u = 0$.

Assume that there exist $p, q \in \mathbb{Z}$ and an element $z = \sum_{i=2}^{\infty} \varepsilon_i \alpha^i \in \mathcal{G}_a$ such that $z + p + q\alpha \in \text{int}(\mathcal{G}_a)$. Thus there is an integer $n_0 \geq 0$ such that for all $n \geq n_0$

$$(1) \quad \sum_{i=2}^n \varepsilon_i \alpha^i + p + q\alpha \in \mathcal{G}_a.$$

Case 1: The set $\{i \geq 2, \varepsilon_i \neq 0\}$ is infinite.

In this case, as $\beta > 1$, then there exists a integer $N \geq n_0$ such that $\sum_{i=2}^N \varepsilon_i \beta^i + p + q\beta > 0$. By item (iv) of Proposition 3.3 we deduce that

$$(2) \quad \sum_{i=2}^N \varepsilon_i \beta^i + p + q\beta = \sum_{i=l}^M d_i \beta^i, \quad \text{where } (d_i)_{l \leq i \leq M} \in E(R), \quad l, M \in \mathbb{Z}.$$

From (1) and (2) we have that $\sum_{i=l}^M d_i \alpha^i = \sum_{i=2}^{\infty} \varepsilon_i \alpha^i \in \mathcal{G}_a$.

Therefore, from item (ii) of Proposition 3.3, we have $e_i = 0$ for all $i > M$. Then,

$$\begin{aligned} \sum_{i=2}^N \varepsilon_i \beta^i + p + q\beta &= \sum_{i=2}^M e_i \beta^i \\ &= \sum_{i=l}^M d_i \beta^i. \end{aligned}$$

According to item (v) of Proposition 3.3, we have

$$\tilde{n}\beta^2 + (r(\tilde{n}) + q)\beta + (s(\tilde{n}) + p) = \tilde{l}\beta^2 + r(\tilde{l})\beta + s(\tilde{l}).$$

where $\tilde{n} = \sum_{i=2}^N \varepsilon_i G'_i$ and $\tilde{l} = \sum_{i=2}^M e_i G'_i$. Therefore, $\tilde{l} = \tilde{n}$ and $\varepsilon_i = e_i$ for all i (by (i) of Proposition 3.3). Thus, $p = q = 0$.

Case 2: The set $\{i \geq 2, \varepsilon_i \neq 0\}$ is finite.

Let $N = \max\{i \geq 2, \varepsilon_i \neq 0\}$. If $\sum_{i=2}^N \varepsilon_i \beta^i + p + q\beta \geq 0$ then we use the same argument than in case 1.

Assume that $\sum_{i=2}^N \varepsilon_i \beta^i + p + q\beta < 0$. We have

$$\sum_{i=2}^N \varepsilon_i \alpha^i + p + q\alpha = \sum_{i=2}^{\infty} d_i \alpha^i \in \mathcal{G}_a \subset \mathcal{R}_a, \text{ where } (d_i)_{i \geq 2} \in E(R).$$

Since $\sum_{i=2}^N \varepsilon_i \alpha^i$ is a interior point of \mathcal{R}_a (item (iii) of Proposition 3.3), then there exists a non-negative integer M such that

$$-p - q\alpha + \sum_{i=2}^M d_i \alpha^i = \sum_{i=2}^{\infty} e_i \alpha^i \in \mathcal{R}_a.$$

Since $-p - q\beta + \sum_{i=2}^M d_i \beta^i > 0$ then $-p - q\beta + \sum_{i=2}^M d_i \beta^i = \sum_{i=l}^K f_i \beta^i$ where $(f_i)_{l \leq i \leq K} \in E(R)$ and $l, K \in \mathbb{Z}$. Therefore,

$$-p - q\alpha + \sum_{i=2}^M d_i \alpha^i = \sum_{i=l}^K f_i \alpha^i = \sum_{i=2}^{\infty} e_i \alpha^i.$$

By item (ii) of Proposition 3.3, we deduce that $e_i = 0$ for all $i > K$ and by the same argument used in case 1, we have that $p = q = 0$. \square

Proposition 3.5. *The boundary of \mathcal{G}_a satisfies the following properties:*

- 1) $\partial \mathcal{G}_a = \bigcup_{u \in A} \mathcal{G}_a \cap (\mathcal{G}_a + u)$ where A is a finite set belonging to $\mathbb{Z} + \alpha\mathbb{Z}$, whose cardinality is even and greater than or equal to 6 and $\{\pm 1, \pm \alpha, \pm(\alpha - 1)\} \subset A$.
- 2) Let $z \in \partial \mathcal{G}_a$ then there exist $(\varepsilon_i)_{i \geq 2}$ and $(\varepsilon'_i)_{i \geq l} \in E(G)$, $l < 2$ such that $z = \sum_{i=2}^{+\infty} \varepsilon_i \alpha^i = \sum_{i=l}^{+\infty} \varepsilon'_i \alpha^i$ and $\varepsilon'_l \neq 0$.

Proof:

- (1) Let $z \in \partial \mathcal{G}_a$, then there exists a sequence $(z_n)_{n \geq 0}$ such that

$$\lim_{n \rightarrow +\infty} z_n = z \text{ and } z_n \notin \mathcal{G}_a, \forall n \geq 0.$$

By Theorem 3.1 (a), there exists a sequence $(p_n)_{n \geq 0}$ of elements $\mathbb{Z} + \alpha\mathbb{Z}$ such that for all $n \geq 0$, $z_n \in \mathcal{G}_a + p_n$. Then $(p_n)_{n \geq 0}$ is bounded. Since $\mathbb{Z} + \alpha\mathbb{Z}$ is a discrete group then there exists a subsequence $(p_{k_n})_{n \geq 0}$ such that for all n , $p_{k_n} = p \in \mathbb{Z} + \alpha\mathbb{Z}$. Since $z_{k_n} \in \mathcal{G}_a + p$, we have $z = \lim z_{k_n} \in \mathcal{G}_a + p$. Therefore,

$$\partial \mathcal{G}_a \subset \bigcup_{p \in \mathbb{Z} + \alpha\mathbb{Z}} \mathcal{G}_a \cap (\mathcal{G}_a + p).$$

On the other hand, if $z \in \mathcal{G}_a \cap (\mathcal{G}_a + p)$, $p \in \mathbb{Z} + \alpha\mathbb{Z} \setminus \{0\}$. Then by Theorem 3.1 (b), $z \notin \text{int}(\mathcal{G}_a)$. Therefore, $z \in \partial \mathcal{G}_a$. Hence, $\partial \mathcal{G}_a = \bigcup_{p \in \mathbb{Z} + \alpha\mathbb{Z}} \mathcal{G}_a \cap (\mathcal{G}_a + p) = \bigcup_{p \in A} \mathcal{G}_a \cap (\mathcal{G}_a + p)$ where $A = \{p \in \mathbb{Z} + \alpha\mathbb{Z}, \mathcal{G}_a \cap (\mathcal{G}_a + p) \neq \emptyset\}$. Since $A \subset \mathbb{Z} + \alpha\mathbb{Z} \cap (\mathcal{G}_a - \mathcal{G}_a)$, we deduce that A is finite set. Finally, the cardinality of A is even because if $u \in A$ then $-u \in A$.

Now, we prove that $\{\pm 1, \pm \alpha, \pm(\alpha - 1)\} \subset A$.

In fact, it's easy to see that $-\alpha^3$ can be written in the following ways:

$$\begin{aligned} -\alpha^3 &= (a-1) \sum_{i=1}^{\infty} (\alpha^{4i+1} + \alpha^{4i+2}) \\ &= \alpha + (a-2)\alpha^3 + (a-1) \sum_{i=1}^{\infty} (\alpha^{4i} + \alpha^{4i+1}) \\ &= 1 + (a-1)\alpha^2 + (a-2)\alpha^3 + (a-1) \sum_{i=1}^{\infty} (\alpha^{4i} + \alpha^{4i+1}). \end{aligned}$$

Hence, $-\alpha^3 \in \mathcal{G}_a \cap (\mathcal{G}_a + \alpha) \cap (\mathcal{G}_a + 1)$. Therefore, 1 and α belong to A . We also show that

$$\begin{aligned} z &= \alpha - 1 + (a-1) \sum_{i=1}^{\infty} (\alpha^{4i} + \alpha^{4i+1}) \\ &= (a-1) \sum_{i=1}^{\infty} (\alpha^{4i-2} + \alpha^{4i+1}) \\ &= \alpha + (a-2)\alpha^2 + \sum_{i=1}^{\infty} (\alpha^{4i-1} + \alpha^{4i+2}). \end{aligned}$$

Then, $z \in \mathcal{G}_a \cap (\mathcal{G}_a + \alpha - 1) \cap (\mathcal{G}_a + \alpha)$. Therefore, $\alpha - 1$ belongs to A .

- (2) Let $z \in \partial \mathcal{G}_a$ then

$$(3) \quad z = n + p\alpha + \sum_{i=2}^{+\infty} \varepsilon_i \alpha^i = \sum_{i=2}^{\infty} \varepsilon'_i \alpha^i,$$

where $(\varepsilon_i)_{i \geq 2}, (\varepsilon'_i)_{i \geq 2} \in E(G)$ and $(n, p) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. We have to consider the following cases:

- (a) If the set $\{i \geq 2, \varepsilon_i \neq 0\}$ is finite, then $\mathcal{G}_a \cap \text{int}(\mathcal{G}_a + n + p\alpha) \neq \emptyset$, which contradicts item b) of Theorem 3.1.
- (b) Assume that the set $\{i \geq 2, \varepsilon_i \neq 0\}$ is infinite. Let $k \geq 2$ be an integer and $z_k = n + p\alpha + \sum_{i=2}^k \varepsilon_i \alpha^i$. We have that $\lim_{k \rightarrow +\infty} z_k = \sum_{i=2}^{+\infty} \varepsilon'_i \alpha^i = z$. On the other hand, there exists an integer $N \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $\tilde{z}_k = n + p\beta + \sum_{i=2}^k \varepsilon_i \beta^i > 0$. Hence $z_k = \sum_{i=l_k}^{N_k} \varepsilon''_{i,k} \alpha^i$, $(\varepsilon''_{i,k}) \in E(G)$ and $\varepsilon''_{l_k,k} > 0$. Moreover, $l_k < 2$, because otherwise $n = p = 0$. On the other hand, there exists $s \in \mathbb{Z}$ such that $s < l_k$ for all integer $k \geq 0$ (because $(z_k)_{k \geq 0}$ is bounded). Then, for all integer $k \geq N$, $z_k \in \bigcup_{t=s}^1 \mathcal{E}_t$ where $\mathcal{E}_t = \{\sum_{i=t}^{+\infty} \varepsilon_i \alpha^i, (\varepsilon_i)_{i \geq t} \in E(G), \varepsilon_t > 0\}$. Since $\bigcup_{t=s}^1 \mathcal{E}_t$ compact, we have $z = \lim_{k \rightarrow +\infty} z_k \in \bigcup_{t=s}^1 \mathcal{E}_t$. Therefore, $z = \sum_{i=l}^{+\infty} \varepsilon''_i \alpha^i$, $(\varepsilon''_i)_{l \leq i \leq +\infty} \in E(G)$, $l < 2$. \square

4. DEFINITION OF THE AUTOMATON RECOGNIZING THE POINTS WITH AT LEAST TWO EXPANSIONS

In this section we proceed to the construction of the automaton \mathcal{A} that characterize the boundary of \mathcal{G}_a and \mathcal{R}_a . The set of states of the automaton \mathcal{A} (see Theorem 4.2) is the set $S_a = \{0, \pm\alpha, \pm\alpha^2, \pm(\alpha - \alpha^2), \pm(1 + (a-1)\alpha^2), \pm(1 + (a-2)\alpha^2), \pm(1 - \alpha + (a-1)\alpha^2), \pm(1 - 2\alpha + a\alpha^2)\}$.

Let s and t be two states. The set of edges is the set of $(s, (e, f), t) \in S \times \{0, 1, \dots, a-1\}^2 \times S$ satisfying $t = \frac{s}{\alpha} + (e - f)\alpha^2$. The set of initial states is $\{(0, (0, 0), 0)\}$.

Let us explain the behaviour of this automaton. Let $\varepsilon = (\varepsilon_i)_{i \geq l}$ and $\varepsilon' = (\varepsilon'_i)_{i \geq l}$ belonging $E(R)$ (resp. $E(G)$), $x = \sum_{i=l}^{\infty} \varepsilon_i \alpha^i$ and $y = \sum_{i=l}^{\infty} \varepsilon'_i \alpha^i$. Suppose $x = y$. For all $k \geq l$ we put

$$(4) \quad A_k(\varepsilon, \varepsilon') = \alpha^{-k+2} \sum_{i=l}^k (\varepsilon_i - \varepsilon'_i) \alpha^i.$$

In the following we prove that all the A_k , $k \in \mathbb{N}$, belong to S . Clearly, for all $k \geq l$,

$$(5) \quad A_{k+1}(\varepsilon, \varepsilon') = \frac{A_k(\varepsilon, \varepsilon')}{\alpha} + (\varepsilon_{k+1} - \varepsilon'_{k+1})\alpha^2.$$

Let s be the smallest integer such that $\varepsilon_s \neq \varepsilon'_s$. Hence $A_i(\varepsilon, \varepsilon') = 0$ for $i \in \{l, \dots, s-1\}$. Suppose $\varepsilon_s > \varepsilon'_s$. Then, $A_s = (\varepsilon'_s - \varepsilon_s)\alpha^2 = \alpha^2$. From (5) we deduce $A_{s+1}(\varepsilon, \varepsilon') = \alpha + (\varepsilon_{s+1} - \varepsilon'_{s+1})\alpha^2$ which should belong to $S_{a,b}$. Hence $A_{s+1}(\varepsilon, \varepsilon') = \alpha$ if $\varepsilon_{s+1} = \varepsilon'_{s+1}$ or $A_{s+1}(\varepsilon, \varepsilon') = \alpha - \alpha^2$ if $(\varepsilon_{s+1}, \varepsilon'_{s+1}) = (t_1, t_1 + 1)$, where $0 \leq t_1 \leq a-2$. Continuing by the same way and using the fact that the set of states S is finite, we obtain a finite state automaton.

Remark 4.1. *The idea of using finite state automaton to recognize points that have at least 2 α -expansions is old. It was done in the case of $\alpha = 1/\gamma$ where $\gamma > 1$ is a Pisot number and the digits belong to a finite set of integer numbers by Frougny in [14]. In [34] Thurston proved the same result in the case where β is a Pisot complex numbers and the digits are in a finite subset of algebraic integers in $\mathbb{Q}(\gamma)$ (see also [17], [24]). The difficulty remains in the fact that it is not easy to find exactly the set of states. The classical method uses the modulus of α . In this work, we give a method which does not use the modulus of α , with this we could find all the states for the automata associated to a class of cubic Pisot unit numbers.*

4.1. Characterization of the points with two expansions. Let $\varepsilon = (\varepsilon_i)_{i \geq l}$ and $\varepsilon' = (\varepsilon'_i)_{i \geq l}$ in $E(R)$ (resp. $E(G)$) where $l \in \mathbb{Z}$, $x = \sum_{i=l}^{\infty} \varepsilon_i \alpha^i$ and $y = \sum_{i=l}^{\infty} \varepsilon'_i \alpha^i$. Suppose that $x = y$. For all $k \geq l$ put

$$(6) \quad A_k(\varepsilon, \varepsilon') = A_k = \alpha^{-k+2} \sum_{i=l}^k (\varepsilon_i - \varepsilon'_i) \alpha^i.$$

Then,

$$(7) \quad A_{k+1} = \frac{A_k}{\alpha} + (\varepsilon_{k+1} - \varepsilon'_{k+1})\alpha^2.$$

For all $\varepsilon = (\varepsilon_i)_{i \geq l}$ and $\varepsilon' = (\varepsilon'_i)_{i \geq l}$ in $E(R)$ (resp. $E(G)$), let

$$S(\varepsilon, \varepsilon') = \{A_k(\varepsilon, \varepsilon'); k \geq l\} = \left\{ \alpha^{-k+2} \sum_{i=l}^k (\varepsilon_i - \varepsilon'_i) \alpha^i; k \geq l \right\}.$$

Theorem 4.2. *Let $x = \sum_{i=l}^{\infty} \varepsilon_i \alpha^i$, $y = \sum_{i=l}^{\infty} \varepsilon'_i \alpha^i$, where $\varepsilon = (\varepsilon_i)_{i \geq l}$ and $\varepsilon' = (\varepsilon'_i)_{i \geq l}$ in $E(R)$ (resp. $E(G)$). Thus, $x = y$ if and only if $S(\varepsilon, \varepsilon')$ is finite. Moreover, $S(\varepsilon, \varepsilon') \subset S_a = \{0, \pm\alpha, \pm\alpha^2, \pm(\alpha - \alpha^2), \pm(1 + (a-1)\alpha^2), \pm(1 + (a-2)\alpha^2), \pm(1 - \alpha + (a-1)\alpha^2), \pm(1 - 2\alpha + a\alpha^2)\}$. And*

$$S_a = \bigcup_{(\varepsilon, \varepsilon') \in \Delta} S(\varepsilon, \varepsilon').$$

where $\Delta = \{((\varepsilon_i)_{i \geq l}, (\varepsilon'_i)_{i \geq l}) \in X \times X; \sum_{i=l}^{\infty} \varepsilon_i \alpha^i = \sum_{i=l}^{\infty} \varepsilon'_i \alpha^i\}$, where $X = E(R)$ (resp. $X = E(G)$).

Proof: If $S(\varepsilon, \varepsilon')$ is finite, then since $0 < |\alpha| < 1$, we have $\sum_{i=l}^{+\infty} \varepsilon_i \alpha^i = \sum_{i=l}^{+\infty} \varepsilon'_i \alpha^i$.

Let $x = \sum_{i=l}^{\infty} \varepsilon_i \alpha^i$ and $y = \sum_{i=l}^{\infty} \varepsilon'_i \alpha^i$ with $\varepsilon = (\varepsilon_i)_{i \geq l}$ and $\varepsilon' = (\varepsilon'_i)_{i \geq l}$ in $E(R)$ (resp. $E(G)$). Assume $x = y$. Then for all $k \geq l$, we have

$$(8) \quad A_k = \sum_{i=k+1}^{\infty} (\varepsilon'_i - \varepsilon_i) \alpha^{i-k+2} = \sum_{i=3}^{\infty} (\varepsilon'_{i+k-2} - \varepsilon_{i+k-2}) \alpha^i \in \alpha\mathcal{R} - \alpha\mathcal{R} \text{ (resp. } \alpha\mathcal{G} - \alpha\mathcal{G}).$$

Let $k \geq l$ and assume that $A_k \neq 0$. By (6) and the fact that α is an algebraic integer of degree 3, we deduce that

$$(9) \quad A_k = n_k \alpha^2 + p_k \alpha + q_k, \text{ where } n_k, p_k, q_k \in \mathbb{Z}.$$

Put $\widetilde{A}_k = n_k \beta^2 + p_k \beta + q_k$. Since \widetilde{A}_k or $-\widetilde{A}_k$ belong to $\mathbb{Z}[\beta] \cap \mathbb{R}^+$, we deduce according to item (iv) of Proposition 3.3, that there exists a sequence $(c_i)_{s_k \leq i \leq m_k} \in E(R)$ such that $c_{m_k} > 0$ and

$$(10) \quad \widetilde{A}_k = n_k \beta^2 + p_k \beta + q_k = \pm \sum_{i=s_k}^{m_k} c_i \beta^i.$$

Assume that $\widetilde{A}_k = \sum_{i=s_k}^{m_k} c_i \beta^i$. By using (6), (9), (10) and the fact that β and α are algebraic conjugates, we get

$$(11) \quad \beta^{-k+2} \sum_{i=l}^k \varepsilon_i \beta^i = \beta^{-k+2} \sum_{i=l}^k \varepsilon'_i \beta^i + \sum_{i=s_k}^{m_k} c_i \beta^i.$$

According to (vi) of Proposition 3.3, we deduce that $\beta^{-k+2} \sum_{i=l}^k \varepsilon_i \beta^i < \beta^3$. Consequently $m_k \leq 2$. Putting $c_i = 0$ for all $i > m_k$, we have $n_k \beta^2 + p_k \beta + q_k = \sum_{i=s_k}^2 c_i \beta^i$. Since β is a Pisot number, using Proposition [2] in [13], we deduce that there exists an integer $s = s(a)$ such that $s \leq s_k$. Therefore,

$$(12) \quad A_k = \sum_{i=s}^2 c_i \alpha^i.$$

Then

$$(13) \quad S_a \subset \left\{ \sum_{i=s}^2 c_i \alpha^i, (c_i)_{s \leq i \leq 2} \in E(R) \right\}.$$

Note that if

$$(14) \quad A_k = \sum_{i=s}^2 c_i \alpha^i \text{ then } \widetilde{A}_k = \sum_{i=s}^2 c_i \beta^i < \beta^3.$$

To prove that $S_a = \{0, \pm\alpha, \pm\alpha^2, \pm(\alpha - \alpha^2), \pm(1 + (a-1)\alpha^2), \pm(1 + (a-2)\alpha^2), \pm(1 - \alpha + (a-1)\alpha^2), \pm(1 - 2\alpha + a\alpha^2)\}$, we need the following important result:

Proposition 4.3. *Let $n, p, q \in \mathbb{Z}$ and $z = n + p\alpha + q\alpha^2$. If $z \in S_a$ then $|n| \leq 1$.*

Proof: Let $S_a = \{A_k = n_k + p_k\alpha + q_k\alpha^2, k \geq 0\}$ and $n = \max\{|n_k|, k \geq 0\}$. Suppose $n \geq 2$ and let $k \in \mathbb{N}$ such that $A_k = n + p\alpha + q\alpha^2$, $p, q \in \mathbb{Z}$. Thus by (7), $A_{k+1} = \frac{n}{\alpha} + p + q\alpha + d\alpha^2$, $|d| \leq a-1$, and using that $\alpha^{-1} = \alpha^2 - a\alpha + 1$ we have

$$A_{k+1} = (p+n) + (q-na)\alpha + (d+n)\alpha^2.$$

Therefore $|p+n| \leq n$ and then $-2n \leq p \leq 0$.

On the other hand, $A_{k+2} = (q-na+p+n) + (d+n-a(p+n))\alpha + (f+p+n)\alpha^2$, where $|f| \leq a-1$, and by the same reason $n(a-2) - p \leq q$. Then we have

$$(15) \quad \begin{aligned} \tilde{A}_k &= n + p\beta + q\beta^2 = 1 - \beta + a\beta^2 + (n-1) + (p+1)\beta + (q-a)\beta^2 \\ &\geq \beta^3 + (n-1) + (p+1)\beta + (n(a-2) - p - a)\beta^2, \end{aligned}$$

and since $n \geq 2$ then $n(a-2) - p - a \geq a-4-p$. Thus we conclude by (15) that

$$(16) \quad \tilde{A}_k \geq \beta^3 + (n-1) + (p+1)\beta + (a-4-p)\beta^2.$$

We have to analyze all the following cases.

1- $a \geq 4$ or ($a = 3$ and $-2n < p < 0$).

Here we have $a-4-p > 0$ and follows by (16) that

$$\tilde{A}_k \geq \beta^3 + (n-1) + (p+1)\beta + (a-4-p)\beta^2 \geq \beta^3 + (n-1) + (p+1+a-4-p)\beta \geq \beta^3.$$

This cannot occur because of (14).

2- If $a = 3$ and $p = 0$.

Here we have $A_{k+3} = (d+q-4n) + (f+7n-3q)\alpha + (g+q-2n)\alpha^2$, $|g| \leq a-1 = 2$, and since $|d+q-4n| \leq n$, $|d| \leq 2$ then $-n \leq d+q-4n \leq 2+q-4n$. Thus

$$(17) \quad 1 \leq 3n-5 \leq q-3.$$

By (15) and $n \geq 2$, we have

$$\tilde{A}_k \geq \beta^3 + (n-1) + \beta + (q-3)\beta^2 > \beta^3.$$

3- If $a = 2$ and $p \leq -3$.

Again $a-4-p > 0$, and by (16) $\tilde{A}_k \geq \beta^3 + (n-1) + (p+1)\beta + (-2-p)\beta^2$. We have $\beta^2 = 2\beta - 1 + \beta^{-1}$, $\beta > 1$ and therefore

$$\begin{aligned} \tilde{A}_k &\geq \beta^3 + (-2-p)\beta^{-1} + (n+1+p) + (-3-p)\beta \\ &> \beta^3 + (-2-p)\beta^{-1} + (n-2) \geq \beta^3. \end{aligned}$$

Therefore if $A_k = n + p\alpha + q\alpha^2$ then

$$(18) \quad a = 2 \text{ and } p \in \{-2, -1, 0\}.$$

4- If $a = 2$ and $-2 \leq p \leq 0$.

Here the possibilities are $A_k = n + q\alpha^2$ or $A_k = n - \alpha + q\alpha^2$ or $A_k = n - 2\alpha + q\alpha^2$.

• If $A_k = n + q\alpha^2$.

We have $A_{k+1} = n + (q-2n)\alpha + (d+n)\alpha^2$ and, using the previous cases, we get $q = 2n$ or $q = 2n-1$ or $q = 2n-2$ and thus $\tilde{A}_k = n + q\beta^2 > \beta^3$.

• $A_k = n - \alpha + q\alpha^2$.

Then

$$\begin{aligned} A_{k+1} &= (n-1) + (q-2n)\alpha + (d+n)\alpha^2 \text{ and} \\ A_{k+2} &= (q-n-1) + (d+n-2(n-1))\alpha + (f+n-1)\alpha^2. \end{aligned}$$

Since $-n \leq q-n-1 \leq n$ then $1 \leq q$.

If $q \geq 2 = a$ then

$$\tilde{A}_k = n - \beta + q\beta^2 \geq n - \beta + a\beta^2 \geq \beta^3.$$

If $q = 1$, then $A_{k+2} = -n + (d+n-2(n-1))\alpha + (f+n-1)\alpha^2$. Since $A_{k+2} \in S_a$, then $-A_{k+2} \in S_a$. Therefore

$$-A_{k+2} = n + (2(n-1) - n - d)\alpha + (-f - n + 1)\alpha^2.$$

Using (18) we get $2(n-1) - n - d \in \{-1, -2\}$.

If $2(n-1) - n - d = -2$, then $d = n \leq a-1 = 1$. That is absurd, since $n \geq 2$.

If $2(n-1) - n - d = -1$, then $d = n-1 \leq 1$, thus $n \leq 2$.

Hence $n = 2$ and thus $A_k = 2 - \alpha + \alpha^2$.
If $A_k = 2 - \alpha + \alpha^2$, then

$$\begin{aligned} A_{k+1} &= 1 + (1 - 2a)\alpha + (d + 2)\alpha^2 \\ &= 1 - 3\alpha + (d + 2)\alpha^2, \end{aligned}$$

and

$$\begin{aligned} A_{k+2} &= -2 + (d + 2 - a)\alpha + (f + 1)\alpha^2 \\ &= -2 + d\alpha + (f + 1)\alpha^2. \end{aligned}$$

We know that $A_{k+2} = -2 + \alpha - \alpha^2$ or $A_{k+2} = -2 + 2\alpha - \alpha^2$. In the first case, we obtain $d = 1$ and $f = -2$, which is impossible because $|f| \leq a - 1 = 1$. In the second case, we get $d = 2$ which is also absurd.

- $A_k = n - 2\alpha + q\alpha^2$.

Then $A_{k+1} = (n - 2) + (q - 2n)\alpha + (d + n)\alpha^2$ and

$$A_{k+2} = (q - 2n + n - 2) + (d + n - 2(n - 2))\alpha + (f + n - 2)\alpha^2, |f| \leq 1.$$

Since $-n \leq q - 2n + n - 2 \leq n$ then $2 \leq q$.

If $q \geq 3$, then

$$\begin{aligned} \tilde{A}_k &= n - 2\beta + q\beta^2 \geq n - 2\beta + (a + 1)\beta^2 \\ &= \beta^3 + (\beta^2 - \beta) > \beta^3. \end{aligned}$$

If $q = 2$, then $A_k = n - 2\alpha + 2\alpha^2$ and $A_{k+2} = -n + (d - n + 4)\alpha + (f + n - 2)\alpha^2$. Since $-A_{k+2} \in S$ we have $d - n + 4 = 2$ and therefore $d = n - 2 \leq 1$ and $n \leq 3$.

If $n = 2$, then $A_k = 2 - 2\alpha + 2\alpha^2$, $A_{k+1} = (2 - 2a)\alpha + (d + 2)\alpha^2 = -2\alpha + (d + 2)\alpha^2$. Then $A_{k+2} = -2 + (d + 2)\alpha + f\alpha^2$, hence $d = 0$ and $f = 2$. That is impossible.

If $n = 3$, then $A_k = 3 - 2\alpha + 2\alpha^2$, $A_{k+1} = 1 + (2 - 3a)\alpha + (d + 3)\alpha^2 = 1 - 4\alpha + (d + 3)\alpha^2$, then $A_{k+2} = -3 + (d + 3 - a)\alpha + (f + 1)\alpha^2$. Hence $d + 3 - a = 2$, thus $d = a - 1 = 1$ and $f + 1 = -2$. Therefore $f = -3$. That is an absurd.

□

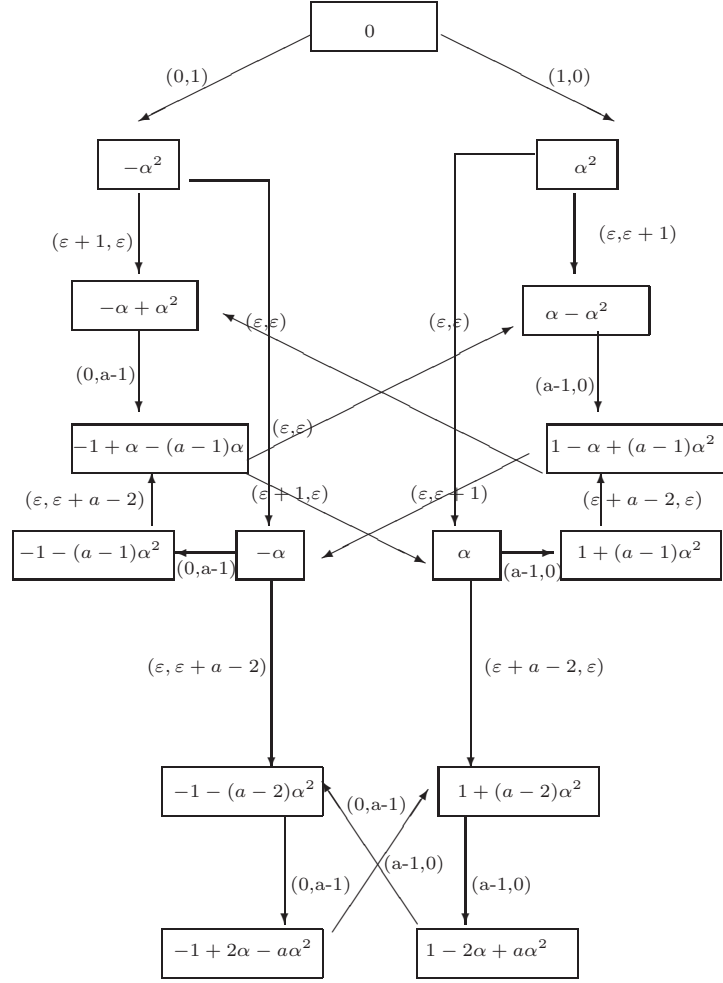
Now we will prove that

$$S_a = \{0, \pm\alpha, \pm\alpha^2, \pm(\alpha - \alpha^2), \pm(1 + (a - 1)\alpha^2), \pm(1 + (a - 2)\alpha^2), \pm(1 - \alpha + (a - 1)\alpha^2), \pm(1 - 2\alpha + a\alpha^2)\}.$$

If $A_k = n + p\alpha + q\alpha^2 \in S_a$, then by proposition 4.3 we have $|n| \leq 1$. Consider the following cases:

- 1- $n = 1$, then we have $A_k = 1 + p\alpha + q\alpha^2$ and $A_{k+1} = (p + 1) + (q - a)\alpha + (d + 1)\alpha^2$, where $|d| \leq a - 1$. Hence by Proposition 4.3, $p \in \{-2, -1, 0\}$.
 - If $p = 0$ then $A_{k+1} = 1 + (q - a)\alpha + (d + 1)\alpha^2$. Thus we have $a - 2 \leq q \leq a$. If $q = a$ then $\tilde{A}_k = 1 + a\beta^2 > \beta^3$, which is impossible because of (14). Hence we have the states $A_k = 1 + (a - 1)\alpha^2$ or $A_k = 1 + (a - 2)\alpha^2$.
 - If $p = -1$ then $A_{k+1} = (q - a)\alpha + (d + 1)\alpha^2$ and $A_{k+2} = (q - a) + (d + 1)\alpha + e\alpha^2$, $|e| \leq a - 1$. Hence we have $-1 \leq q - a \leq 1$. For the cases $q = a$ or $q = a + 1$ we have $\tilde{A}_k \geq \beta^3$. Hence we get the state $A_k = 1 - \alpha + (a - 1)\alpha^2$.
 - If $p = -2$, then $A_{k+1} = -1 + (q - a)\alpha + (d + 1)\alpha^2$ and $A_{k+2} = (q - a - 1) + (d + 1 + a)\alpha + e\alpha^2$, $|e| \leq a - 1$. Hence $-1 \leq q - a - 1 \leq 1$. If $q = a$ we get the state $A_k = 1 - 2\alpha + a\alpha^2$. If $q = a + 1$ or $q = a + 2$, then $\tilde{A}_k > \beta^3$.
- 2- $n = 0$, then we have $A_k = p\alpha + q\alpha^2$ and $A_{k+1} = p + q\alpha + d\alpha^2$. Hence by Proposition 4.3, $p \in \{-1, 0, 1\}$.
 - If $p = 0$ then $A_{k+1} = p + q\alpha + d\alpha^2$, $A_{k+2} = q + d\alpha + e\alpha^2$ and by Proposition 4.3, we deduce that $q = 0, \pm 1$. Therefore, $A_k = 0, \pm\alpha^2$.
 - If $p = 1$ then $A_{k+1} = 1 + q\alpha + d\alpha^2$ and $q \in \{0, -1, -2\}$. Therefore, $A_k = \alpha, \alpha - \alpha^2, \alpha - 2\alpha^2$. If $A_k = \alpha - 2\alpha^2$ is a state then $A_{k+1} = 1 - 2\alpha + d\alpha^2$ is also a state. But as noted earlier $A_{k+1} = 1 - 2\alpha + d\alpha^2 = 1 - 2\alpha + a\alpha^2$ and, therefore $d = a$, which is impossible since $d \leq a - 1$.
 - If $p = -1$. Using the same ideas of previous case, we obtain the states $A_k = -\alpha, -\alpha + \alpha^2$.

□

Automaton \mathcal{A}

As a Corollary we obtain the following result:

Theorem 4.4. *For all $a \geq 2$ we have*

$$\partial \mathcal{G}_a = \bigcup_{u \in B} \mathcal{G}_a \cap (\mathcal{G}_a + u)$$

where $B = \{\pm 1, \pm \alpha, \pm(\alpha - 1)\}$.

Proof: According to item (1) of Proposition 3.5, we have that $\bigcup_{u \in B} \mathcal{G}_a \cap (\mathcal{G}_a + u) \subset \partial \mathcal{G}_a$. If $z \in \partial \mathcal{G}_a$ then by item (2) of Proposition 3.5 we have

$$(19) \quad z = \sum_{i=2}^{\infty} \varepsilon_i \alpha^i = \sum_{i=l}^{\infty} \varepsilon'_i \alpha^i \text{ where } l < 2 \text{ and } \varepsilon'_l \neq 0.$$

Then $r = (\varepsilon'_l, 0)(\varepsilon'_{l+1}, 0) \dots (\varepsilon'_1, 0)(\varepsilon'_2, \varepsilon_2) \dots$, $l < 2$ is a path in the automaton \mathcal{A} starting from the initial state. Therefore,

$$(20) \quad r = (1, 0)(x_0, x_0 + 1)(a - 1, 0) \dots$$

or

$$(21) \quad r = (1, 0)(x_0, x_0)(a - 1, 0)(x_1 + a - 2, x_1)(x_2, x_2)(0, a - 1) \dots$$

or

$$(22) \quad r = (1, 0)(x_0, x_0)(a - 1, 0)(x_1 + a - 2, x_1)(x_2, x_2 + 1) \dots$$

or

$$(23) \quad r = (1, 0)(x_0, x_0)(x_1 + a - 2, x_1)yyy \text{ where } y = (a - 1, 0)(0, a - 1)(0, a - 1)(a - 1, 0)$$

where $x_0, x_1, x_2 \in \{0, 1, \dots, a - 1\}$.

1- If $r = (1, 0)(x_0, x_0 + 1)(a - 1, 0) \dots$ then by (19) we have $l = 1$ and

$$(24) \quad z = (x_0 + 1)\alpha^2 + \sum_{i=4}^{\infty} \varepsilon_i \alpha^i = \alpha + x_0 \alpha^2 + (a - 1)\alpha^3 + \sum_{i=4}^{\infty} \varepsilon'_i \alpha^i.$$

Therefore $z \in \mathcal{G}_a \cap (\mathcal{G}_a + \alpha)$.

2- If $r = (1, 0)(x_0, x_0)(a - 1, 0)(x_1 + a - 2, x_1)(x_2, x_2)(0, a - 1) \dots$, then $x_1 = 0$ and we have the following possibilities:

2.1- If $x_0 > 0$ then $l = 1$ and $z \in \mathcal{G}_a \cap (\mathcal{G}_a + \alpha)$.

2.2- If $x_0 = 0$ and $x_2 > 0$ then $l \in \{-2, -1, 0, 1\}$. If $l = 0$ or 1 , then $z \in \mathcal{G}_a \cap (\mathcal{G}_a + \alpha^l)$.

If $l = -1$, then we have by equation (19)

$$\begin{aligned} z &= x_2 \alpha^3 + (a - 1)\alpha^4 + \sum_{i=4}^{\infty} \varepsilon_i \alpha^i \\ &= \frac{1}{\alpha} + (a - 1)\alpha + (a - 2)\alpha^2 + \sum_{i=3}^{\infty} \varepsilon'_i \alpha^i \\ &= 1 - \alpha + (a - 1)\alpha^2 + \sum_{i=3}^{\infty} \varepsilon'_i \alpha^i. \end{aligned}$$

Hence $z \in \mathcal{G}_a \cap (\mathcal{G}_a - \alpha + 1)$.

If $l = -2$, then we have by equation (19)

$$\begin{aligned} z &= x_2 \alpha^2 + (a - 1)\alpha^3 + \sum_{i=4}^{\infty} \varepsilon_i \alpha^i \\ &= \frac{1}{\alpha^2} + (a - 1) + (a - 2)\alpha + x_2 \alpha^2 + \sum_{i=3}^{\infty} \varepsilon'_i \alpha^i \\ &= -\alpha + (x_2 + 1)\alpha^2 + \sum_{i=3}^{\infty} \varepsilon'_i \alpha^i. \end{aligned}$$

Therefore, $z \in \mathcal{G}_a \cap (\mathcal{G}_a - \alpha)$.

2.3- If $x_0 = x_2 = 0$ then $l \in \{-3, -2, -1, 0, 1\}$. If $l = -3$, then

$$z = (a - 1)\alpha^2 + \sum_{i=3}^{\infty} \varepsilon_i \alpha^i = \frac{1}{\alpha^3} + \frac{(a - 1)}{\alpha} + (a - 2) + \sum_{i=2}^{\infty} \varepsilon'_i \alpha^i.$$

Since $\frac{1}{\alpha^3} = (1 - a) + \alpha + \frac{(1 - a)}{\alpha}$, we have $z \in \mathcal{G}_a \cap (\mathcal{G}_a + (\alpha - 1))$.

3- If $r = (1, 0)(x, x)(a - 1, 0)(x_1 + a - 2, x_1)(x_2, x_2 + 1) \dots$, then we can prove by the same way than the previous cases that $z \in \mathcal{G}_a \cap (\mathcal{G}_a + \alpha)$ or $z \in \mathcal{G}_a \cap (\mathcal{G}_a + (1 - \alpha))$ or $z \in \mathcal{G}_a \cap (\mathcal{G}_a - \alpha)$.

4 Finally considering the path: $r = (1, 0)(x, x)(x_1 + a - 2, x_1)sss$ where

$s = (a - 1, 0)(0, a - 1)(0, a - 1)(a - 1, 0)$ and analyzing all possible cases, we have: $z \in \mathcal{G}_a \cap (\mathcal{G}_a + \alpha)$ or $z \in \mathcal{G}_a \cap (\mathcal{G}_a + 1)$ or $z \in \mathcal{G}_a \cap (\mathcal{G}_a - 1)$. \square

5. RAUZY FRACTAL \mathcal{R}_a

Now, let us consider the classical Rauzy fractal \mathcal{R}_a associated to the Pisot unit number $\beta > 1$ satisfying $\beta^3 - a\beta^2 + \beta - 1$. We have

$$\mathcal{R}_a = \left\{ \sum_{i=2}^{\infty} \varepsilon_i \alpha^i \mid \forall i \geq 2, \varepsilon_i = 0, 1, \dots, a - 1, \varepsilon_i \varepsilon_{i-1} \varepsilon_{i-2} \varepsilon_{i-3} <_{lex} (a - 1)(a - 1)01, \text{ where } \varepsilon_1 = \varepsilon_0 = \varepsilon_{-1} = 0 \right\}.$$

As we mentioned before, the set \mathcal{R}_a is a compact, connected subset of \mathbb{C} with interior simply connected. Moreover, \mathcal{R}_a induces a periodic tiling of the plane \mathbb{C} .

Proposition 5.1. \mathcal{R}_a induces a periodic tiling of the plane \mathbb{C} modulo $\mathbb{Z}u + \mathbb{Z}\alpha u$ where $u = \alpha - 1$. Moreover $\partial\mathcal{R}_a = \bigcup_{v \in B} \mathcal{R}_a \cap (\mathcal{R}_a + v)$, where $B = \{\pm u, \pm\alpha u, (1+\alpha)u, (1-\alpha)u\}$ and $\mathcal{R}_a \cap (\mathcal{R}_a + (1+\alpha)u) = \{-1\}$, $\mathcal{R}_a \cap (\mathcal{R}_a + (\alpha-1)u) = \{-\alpha\}$.

Proof: Consider the sequence $(R'_n)_{n \geq 0}$ by $R'_0 = 0$, $R'_1 = 0$, $R'_2 = 1$, $R'_3 = a$, $R'_4 = a^2$, $R'_{n+3} = aR'_{n+2} - R'_{n+1} + R'_n$, $n \geq 2$. Then $R_n = R'_{n+2}$ for all integer $n \in \mathbb{N}$.

On the other hand, we can prove by induction on n that $G'_n = R'_n - G'_{n-2}$ for all integer $n \geq 2$. Thus, using item (vii) of Proposition 3.3, we have

$$\alpha^n = R'_n \alpha^2 - G'_{n-2}(\alpha^2 - \alpha) - G'_{n-1}(\alpha - 1), \forall n \geq 2.$$

Thus $(\varepsilon_i)_{2 \leq i \leq N} \in E(R)$, we have

$$\sum_{i=2}^N \varepsilon_i \alpha^i = n\alpha^2 + p_n(\alpha^2 - \alpha) + q_n(\alpha - 1)$$

where $n = \sum_{i=2}^N \varepsilon_i R'_i$, $p_n = -\sum_{i=2}^N \varepsilon_i G'_{i-2}$ and $q_n = -\sum_{i=2}^N \varepsilon_i G'_{i-1}$. Using item (v) of Proposition 3.3, we deduce that if x, y are the coordinates of α^2 in base $(\alpha^2 - \alpha, \alpha - 1)$, then 1, x and y are \mathbb{Q} -Linearly independent. Hence by Kronecker's Theorem, the set $\{n\alpha^2 + p(\alpha^2 - \alpha) + q(\alpha - 1), n \in \mathbb{N}, p, q \in \mathbb{Z}\}$ is a dense set in \mathbb{C} . Using the fact that \mathcal{R}_a is the closure of the set $\{\sum_{i=2}^N \varepsilon_i \alpha^i, (\varepsilon_i)_{2 \leq i \leq N} \in E(R)\}$ and the same proof of Proposition 3.3, we deduce that $\mathbb{C} = \bigcup_{v \in \mathbb{Z}[\alpha-1] + \mathbb{Z}[\alpha^2-\alpha]} \mathcal{R} \cap (\mathcal{R} + v)$ and if $(\text{int}(\mathcal{R}) + v) \cap \mathcal{R} \neq \emptyset$ where $v \in H = \mathbb{Z}[\alpha-1] + \mathbb{Z}[\alpha^2-\alpha]$ then $v = 0$.

On the other hand, the boundary of \mathcal{R}_a is given by $\partial\mathcal{R}_a = \bigcup_{v \in H - \{0\}} \mathcal{R}_a \cap (\mathcal{R}_a + v)$. Let $w \in S_a = \{0, \pm\alpha, \pm\alpha^2, \pm(\alpha - \alpha^2), \pm(1 + (a-1)\alpha^2), \pm(1 + (a-2)\alpha^2), \pm(1 - \alpha + (a-1)\alpha^2), \pm(1 - 2\alpha + a\alpha^2)\}$ be a state of the automaton \mathcal{A} . From Proposition 4.2 and relation (8), we have $\mathcal{R}_a \cap (\mathcal{R}_a + w/\alpha) \neq \emptyset$. We remove the states $w = 0, \pm\alpha, \pm\alpha^2, \pm(1 + (a-1)\alpha^2)$, because in this cases $w/\alpha \notin G$. Since B is equal to the set of w/α such that $w \in \{\pm(\alpha - \alpha^2), \pm(1 + (a-2)\alpha^2), \pm(1 - \alpha + (a-1)\alpha^2), \pm(1 - 2\alpha + a\alpha^2)\} = \pm(\alpha^3 - \alpha^2), \pm(1 - 2\alpha + a\alpha^2)$, we obtain that $\bigcup_{v \in B} \mathcal{R}_a \cap (\mathcal{R}_a + v) \subset \partial\mathcal{R}_a$.

Now, let z be an element of \mathcal{R}_a . Considering $\{\alpha^2 - \alpha, \alpha - 1\}$ instead of $\{1, \alpha\}$ and using exactly the same argument done in the proof of item 2 of Proposition 3.5, we deduce that there exist $(\varepsilon_i)_{i \geq 2}$ and $(\varepsilon'_i)_{i \geq l} \in E(R)$, $l < 2$ such that $z = \sum_{i=2}^{+\infty} \varepsilon_i \alpha^i = \sum_{i=l}^{+\infty} \varepsilon'_i \alpha^i$ and $\varepsilon'_l \neq 0$. Hence $r = (\varepsilon'_l, 0)(\varepsilon'_{l+1}, 0) \dots (\varepsilon'_1, 0)(\varepsilon'_2, \varepsilon_2) \dots$, $l < 2$ is a path in the automaton \mathcal{A} starting from the initial state. Therefore r satisfies one of the relations (20), (21), (22), (23).

If $r = (1, 0)(x_0, x_0 + 1)(a - 1, 0)(x_1 + a - 2, x_1)(x_2, x_2 + 1) \dots$, then $x_1 = 0$ and $z = (x_0 + 1)\alpha^2 + \sum_{i=4}^{\infty} \varepsilon_i \alpha^i = \alpha + x_0\alpha^2 + (a - 1)\alpha^3 + \sum_{i=4}^{\infty} \varepsilon'_i \alpha^i = -(\alpha^2 - \alpha) + (x_0 + 1)\alpha^2 + (a - 1)\alpha^3 + \sum_{i=4}^{\infty} \varepsilon'_i \alpha^i$. Therefore $z \in \mathcal{R}_a \cap (\mathcal{R}_a - (\alpha^2 - \alpha))$.

If $r = (1, 0)(x_0, x_0)(a - 1, 0)(x_1 + a - 2, x_1)(x_2, x_2)(0, a - 1) \dots$ then, if $x_0 > 0$, we deduce that $z \in \mathcal{R}_a \cap (\mathcal{R}_a - (\alpha^2 - \alpha))$.

If $x_0 = 0$ and $x_2 > 0$, we have $l \in \{-2, -1, 0, 1\}$. If $l = -2$, then $z \in \mathcal{R} \cap (\mathcal{R} - (\alpha^2 - \alpha))$. If $l = -1$, $z \in \mathcal{R} \cap (\mathcal{R} - (\alpha - \alpha))$. If $l = 0$, then $z \in \mathcal{R} \cap (\mathcal{R} - (\alpha - 1))$. If $l = 1$, then $z \in \mathcal{R}_a \cap (\mathcal{R}_a - (\alpha^2 - \alpha))$.

If $x_0 = x_2 = 0$, then $l \in \{-3, -2, -1, 0, 1\}$. If $l = -3$, then $z \in \mathcal{R}_a \cap (\mathcal{R}_a + (\alpha - 1))$.

The cases $r = (1, 0)(x, x)(a - 1, 0)(x_1 + a - 2, x_1)(x_2, x_2 + 1) \dots$ and $r = (1, 0)(x, x)(x_1 + a - 2, x_1)sss$ where $s = (a - 1, 0)(0, a - 1)(0, a - 1)(a - 1, 0)$ are left to the reader.

Using the automaton \mathcal{A} , we deduce that $\mathcal{R}_a \cap (\mathcal{R}_a + (1 + \alpha)u) = \{-1\}$, $\mathcal{R}_a \cap (\mathcal{R}_a + (\alpha - 1)u) = \{-\alpha\}$. Indeed, if $z \in \mathcal{R}_a \cap (\mathcal{R}_a + (1 + \alpha)u) = \mathcal{R}_a \cap (\mathcal{R}_a + \alpha^{-2} + \alpha)$, then the path representing z in the automaton is

$$(1, 0)(0, 0)(0, 0)(1, 0)(1, 0)(0, 1)ttt \dots,$$

where $t = (1, 0)(1, 0)(0, 1)(0, 1)$. Hence, $z = -1$.

The case $\mathcal{R}_a \cap (\mathcal{R}_a + (\alpha - 1)u) = \mathcal{R}_a \cap (\mathcal{R}_a + \alpha^{-1})$ is left to the reader. \square

6. PARAMETRIZATION OF THE BOUNDARY OF \mathcal{R}_2

In this section, for simplicity, we consider the case $a = 2$ and we give a complete description about the boundary of \mathcal{R}_2 where

$$\mathcal{R}_2 = \left\{ \sum_{i=2}^{\infty} \varepsilon_i \alpha^i, \varepsilon_i \in \{0, 1\} \forall i \geq 2, \varepsilon_i \varepsilon_{i-1} \varepsilon_{i-2} \varepsilon_{i-3} <_{lex} 1101 \right\}.$$

We have seen (Proposition 5.1) that $\partial \mathcal{R}_2 = \bigcup_{v \in B} \mathcal{R}_2 \cap (\mathcal{R}_2 + v)$ where $B = \{\pm(\alpha^{-3} + \alpha^{-1}) = \pm(\alpha - 1), \pm\alpha^{-1}, \pm(1 + \alpha^{-2}), \pm(\alpha + \alpha^{-2})\}$. Since $\mathcal{R}_2 \cap (\mathcal{R}_2 \pm v)$ is a point if $v = \alpha^{-2} + \alpha$ or $v = \alpha^{-1}$. We will study the others four regions $\mathcal{R}_v = \mathcal{R}_2 \cap (\mathcal{R}_2 + v)$ where $v \in \{\pm(\alpha - 1), \pm(1 + \alpha^{-2})\}$, in particular, we will prove the following result.

Proposition 6.1. *Let g and h_i , $i = 0, 1, 2$ be the functions defined by $g(z) = \alpha - 1 + \alpha z$, $h_0(z) = \alpha - 1 + \alpha^2 z$, $h_1(z) = -1 + \alpha^3 z$ and $h_2(z) = \alpha^2 + \alpha^3 + \alpha^4 z$ for all $z \in \mathbb{C}$. Then we have the following properties:*

- (a) $\mathcal{R}_{\alpha-1} = g(\mathcal{R}_{\alpha^{-2}+1})$.
- (b) $\mathcal{R}_{\alpha-1} = h_0(\mathcal{R}_{\alpha-1}) \cup h_1(\mathcal{R}_{\alpha-1}) \cup h_2(\mathcal{R}_{\alpha-1})$.
- (c) $h_1(\mathcal{R}_{\alpha-1}) \cap h_2(\mathcal{R}_{\alpha-1}) = \{-1 - \alpha^2 - \alpha^4\} = \{h_2(-\alpha - \alpha^{-1})\}$.
- (d) $h_1(\mathcal{R}_{\alpha-1}) \cap h_0(\mathcal{R}_{\alpha-1}) = \{-1 - \alpha^3\} = \{h_1(-1)\}$.
- (e) $h_0(\mathcal{R}_{\alpha-1}) \cap h_2(\mathcal{R}_{\alpha-1}) = \emptyset$.

Remark 6.2. *Using b), c), d) and e) of the last Proposition, we will construct an explicit continuous and bijective application from $[0, 1]$ to $\mathcal{R}_{\alpha-1}$. Using this fact and a), we obtain an explicit homeomorphism between the circle and the boundary of \mathcal{R}_2 .*

Lemma 6.3. *The following properties are valid:*

- (a) $\mathcal{R}_{\alpha^{-3}+\alpha^{-1}} \cap \mathcal{R}_{1+\alpha^{-2}} = \{-1\}$.
- (b) $\mathcal{R}_{\alpha^{-3}+\alpha^{-1}} \cap \mathcal{R}_{-1-\alpha^{-2}} = \{-\alpha - \alpha^{-1}\}$.
- (c) $\mathcal{R}_{\alpha^{-1}} \cap \mathcal{R}_{1+\alpha^{-2}} = \{-\alpha\}$.

Remark 6.4. *For the proof of Lemma 6.3, we will use the following relations:*

$$(25) \quad \forall n \in \mathbb{Z}, \alpha^n = \alpha^{n-1} + \alpha^{n-2} + \alpha^{n-4}, \alpha^n + \alpha^{n-2} = 2\alpha^{n-1} + \alpha^{n-3}.$$

Proof:

- 1) Let z be an element of $\mathcal{R}_{\alpha^{-3}+\alpha^{-1}} \cap \mathcal{R}_{1+\alpha^{-2}}$. The path representing z in the automaton is $(1,0)(0,1)(1,0)(0,1)(0,0)(0,1)tttt\dots$ where $t = (1,0)(1,0)(0,1)(0,1)$. Then

$$z = \alpha^{-3} + \alpha^{-1} + \sum_{i=1}^{\infty} (\alpha^{4i-1} + \alpha^{4i}).$$

By (25), we have $z + \alpha = \alpha^{-3} + \alpha^{-1}$, then $z = -1$.

- 2) Let z in $\mathcal{R}_{\alpha^{-3}+\alpha^{-1}} \cap \mathcal{R}_{-1-\alpha^{-2}}$. As $\alpha^{-3} + \alpha^{-1} + \alpha^{-2} + 1 = \alpha^{-2} + \alpha$, then $z + 1 + \alpha^{-2} \in \mathcal{R}_{\alpha+\alpha^{-2}}$. The path representing $z + 1 + \alpha^{-2}$ in the automaton is $(1,0)(0,0)(0,0)(1,0)tttt\dots$ where $t = (0,1)(0,1)(1,0)(1,0)$. Then

$$z + 1 + \alpha^{-2} = \sum_{i=1}^{\infty} (\alpha^{4i-2} + \alpha^{4i-1}).$$

By (25), we get $z + 1 + \alpha^{-2} = -1$, then $z = -2 - \alpha^{-2} = -\alpha - \alpha^{-1}$.

- 3) If z is an element of $\mathcal{R}_{\alpha^{-1}} \cap \mathcal{R}_{1+\alpha^{-2}}$, then $z \in \mathcal{R}_{\alpha^{-1}}$ and the path representing z in the automaton is $(1,0)(0,0)(0,0)(1,0)tttt\dots$ where $t = (0,1)(0,1)(1,0)(1,0)$. Then

$$z = \sum_{i=1}^{\infty} (\alpha^{4i-1} + \alpha^{4i}).$$

By (25) we get $z + \alpha = 0$, that is, $z = -\alpha$. □

Proof of Proposition 6.1

a) Let $z \in \mathcal{R}_{\alpha^{-2}+1}$, then, using the automaton we have $z = \alpha^{-2} + 1 + \sum_{i \geq 2} a_i \alpha^i = \sum_{i \geq 2} b_i \alpha^i$ where $a_3 = 0$. Then

$$g(z) = \alpha - 1 + \sum_{i \geq 2} b_i \alpha^{i+1} \in \mathcal{R} + \alpha - 1$$

and

$$g(z) = \alpha^{-1} + 2\alpha - 1 + \sum_{i \geq 2} a_i \alpha^{i+1} = \alpha^2 + \sum_{i \geq 2} a_i \alpha^{i+1} \in \mathcal{R}.$$

We conclude that $g(\mathcal{R}_{\alpha^{-2}+1}) \subseteq \mathcal{R}_{\alpha-1}$.

Now given $w \in \mathcal{R}_{\alpha-1}$, using the automaton, we have

$$w = \alpha - 1 + \sum_{i \geq 3} a_i \alpha^i = \alpha^2 + \sum_{i \geq 3} b_i \alpha^i, \text{ where } b_4 = 0,$$

we have $w = g(z)$, $z \in \mathcal{R}_{\alpha^{-2}+1}$ where $z = \alpha^{-2} + 1 + \sum_{i \geq 3} b_i \alpha^{i-1} = \sum_{i \geq 3} a_i \alpha^{i-1}$.

We conclude that $\mathcal{R}_{\alpha-1} \subseteq g(\mathcal{R}_{\alpha^{-2}+1})$ and then $\mathcal{R}_{\alpha-1} = g(\mathcal{R}_{\alpha^{-2}+1})$.

Let z be an element of $\mathcal{R}_{\alpha-1}$ using the automaton we conclude that $z = \alpha - 1 + \sum_{i \geq 3} a_i \alpha^i$ and $z = \alpha^2 + \sum_{i \geq 3} b_i \alpha^i$.

b) Using (25), we have

$$(26) \quad h_0(z) = \alpha^2 + \sum_{i \geq 3} a_i \alpha^{i+2} = \alpha - 1 + \alpha^4 + \sum_{i \geq 3} b_i \alpha^{i+2}.$$

$$(27) \quad h_1(z) = \alpha^2 + \sum_{i \geq 3} a_i \alpha^{i+3} = \alpha - 1 + \alpha^3 + \alpha^4 + \sum_{i \geq 3} b_i \alpha^{i+3}.$$

$$(28) \quad h_2(z) = h_2(\alpha - 1 + \sum_{i \geq 3} a_i \alpha^i) = \alpha - 1 + \alpha^3 + \alpha^4 + \sum_{i \geq 3} a_i \alpha^{i+4},$$

$$h_2(z) = h_2(\alpha^2 + \sum_{i \geq 3} b_i \alpha^i) = \alpha^2 + \alpha^3 + \alpha^6 + \sum_{i \geq 3} b_i \alpha^{i+4}.$$

Therefore $h_i(\mathcal{R}_{\alpha-1}) \subset \mathcal{R}_{\alpha-1}$, $\forall i \in \{0, 1, 2\}$ and then

$$h_0(\mathcal{R}_{\alpha-1}) \cup h_1(\mathcal{R}_{\alpha-1}) \cup h_2(\mathcal{R}_{\alpha-1}) \subseteq \mathcal{R}_{\alpha-1}.$$

Now take $z \in \mathcal{R}_{\alpha-1}$.

If $(a_3, b_3) = (0, 0)$ then $(a_4, b_4) = (1, 0)$ and

$$z = \alpha - 1 + \alpha^4 + \sum_{i \geq 5} a_i \alpha^i = \alpha^2 + \sum_{i \geq 5} b_i \alpha^i \text{ with } a_7 a_6 a_5 1 <_{lex} 1101.$$

Using (26), we get $z = h_0(z_0)$ where z_0 is the element of $\mathcal{R}_{\alpha-1}$ given by

$$z_0 = \alpha - 1 + \sum_{i \geq 5} b_i \alpha^{i-2} = \alpha^2 + \sum_{i \geq 5} a_i \alpha^{i-2}.$$

If $(a_3, b_3) = (1, 0)$ then $(a_4, b_4)(a_5, b_5) = (1, 0)(0, 0)$ and

$$z = \alpha - 1 + \alpha^3 + \alpha^4 + \sum_{i \geq 6} a_i \alpha^i = \alpha^2 + \sum_{i \geq 6} b_i \alpha^i \text{ with } a_8 a_7 a_6 1 <_{lex} 1101.$$

By (27) we get $z = h_1(z_0)$ where z_0 is the element of $\mathcal{R}_{\alpha-1}$ given by

$$z_0 = \alpha - 1 + \sum_{i \geq 6} b_i \alpha^{i-3} = \alpha^2 + \sum_{i \geq 6} a_i \alpha^{i-3}.$$

If $(a_3, b_3) = (1, 1)$ then $(a_4, b_4)(a_5, b_5)(a_6, b_6) = (1, 0)(0, 0)(0, 1)$ and

$$z = \alpha - 1 + \alpha^3 + \alpha^4 + \sum_{i \geq 7} a_i \alpha^i = \alpha^2 + \alpha^3 + \alpha^6 + \sum_{i \geq 7} b_i \alpha^i \text{ with } b_9 b_8 b_7 1 <_{lex} 1101.$$

By (28), we have $z = h_2(z_0)$ where z_0 is the element of $\mathcal{R}_{\alpha-1}$ given by

$$z_0 = \alpha - 1 + \sum_{i \geq 7} a_i \alpha^{i-4} = \alpha^2 + \sum_{i \geq 7} b_i \alpha^{i-4}.$$

Therefore $\mathcal{R}_{\alpha-1} \subseteq h_0(\mathcal{R}_{\alpha-1}) \cup h_1(\mathcal{R}_{\alpha-1}) \cup h_2(\mathcal{R}_{\alpha-1})$.

c) Let $z \in h_1(\mathcal{R}_{\alpha-1}) \cap h_2(\mathcal{R}_{\alpha-1})$. Then there exist $z_1, z_2 \in \mathcal{R}_{\alpha-1}$ such that $h_1(z_1) = -1 + \alpha^3 z_1 = \alpha^2 + \alpha^3 + \alpha^4 z_2 = h_2(z_2)$. Then $z_1 = \alpha + \alpha z_2 = -1 - \alpha^{-2} + \alpha^2 + \alpha z_2$. Since $z_2 \in \mathcal{R}_{\alpha-1}$, using the automaton, $z_2 = \alpha^2 + \sum_{i=3}^{+\infty} b_i \alpha^i$, where $b_4 = 0$. Therefore

$$z_1 = -1 - \alpha^{-2} + \alpha^2 + \alpha z_2 = -1 - \alpha^{-2} + \alpha^2 + \alpha^3 + \sum_{i \geq 3} b_i \alpha^{i+1}.$$

Hence $z_1 \in \mathcal{R}_{\alpha-1} \cap \mathcal{R}_{-1-\alpha^{-2}} = \{-\alpha^{-1} - \alpha\}$ and then

$$z = h_1(z_1) = -1 - \alpha^2 - \alpha^4.$$

d) Let $z \in h_0(\mathcal{R}_{\alpha-1}) \cap h_1(\mathcal{R}_{\alpha-1})$. Then there exist $z_0, z_1 \in \mathcal{R}_{\alpha-1}$ such that $h_0(z_0) = \alpha - 1 + \alpha^2 z_0 = -1 + \alpha^3 z_1 = h_1(z_1)$ and then we have $z_1 = \alpha^{-2} + \alpha^{-1} z_0$. Since $z_0 \in \mathcal{R}_{\alpha-1}$, then $z_0 = \alpha^2 + \sum_{i \geq 3} b_i \alpha^i$ and

$$z_1 = \alpha^{-2} + \alpha^{-1} z_0 = \alpha^{-2} + \alpha^{-1} (\alpha^2 + \sum_{i \geq 3} b_i \alpha^i) = \alpha^{-2} + \alpha + \sum_{i \geq 3} b_i \alpha^{i-1} \in \mathcal{R}_{\alpha^{-2}+\alpha}.$$

Therefore $z_1 \in \mathcal{R}_{\alpha^{-2}+\alpha} \cap \mathcal{R}_{\alpha-1} = \{-1\}$. Then

$$z = h_1(z_1) = h_1(-1) = -1 - \alpha^3.$$

e) Left to the reader, can be done by the same manner that the others items. □

Lemma 6.5. Consider h_2 as in Proposition 6.1 and $z \in \mathbb{C}$. Then $\lim_{n \rightarrow \infty} h_2^n(z) = -1$.

Proof: Since $h_2(z) = \alpha^2 + \alpha^3 + \alpha^4 z$, we can prove by induction that $h_2^n(z) = \sum_{i=0}^{n-1} (\alpha^{4i+2} + \alpha^{4i+3}) + \alpha^{4n} z$ for all integer $n \geq 1$. Hence

$$\lim_{n \rightarrow \infty} h_2^n(z) = \sum_{i=0}^{\infty} (\alpha^{4i+2} + \alpha^{4i+3}) = \frac{\alpha^2 + \alpha^3}{1 - \alpha^4} = -1. \quad \square$$

6.0.1. *Parametrization of $\mathcal{R}_{\alpha-1}$.* Here, we will give an explicit parametrization of $\mathcal{R}_{\alpha-1}$ and hence for the boundary $\partial \mathcal{R}$. Let z be an element of $\mathcal{R}_{\alpha-1}$. Using Proposition 6.1, there exists a sequence $(z_n)_{n \geq 1}$ in $\mathcal{R}_{\alpha-1}$, such that

$$z = h_{a_1} \circ h_{a_2} \circ \dots \circ h_{a_n}(z_n), \forall n \geq 1.$$

If x is an element of $\mathcal{R}_{\alpha-1}$, the sequence $y_n = h_{a_1} \circ h_{a_2} \circ \dots \circ h_{a_n}(x)$ converges to z because the functions $h_i, i = 0, 1, 2$ are contractions.

Taking $x_0 \in \mathcal{R}_{\alpha-1}$, $t \in [0, 1]$, $t = \sum_{i=1}^{\infty} a_i 3^{-i}$, $a_i \in \{0, 1, 2\}$ we can define a function $f : [0, 1] \rightarrow \mathcal{R}_{\alpha-1}$ by $f(t) = \lim_{n \rightarrow \infty} h_{b_1} \circ \dots \circ h_{b_n}(x_0)$ where $(b_i)_{i \geq 1} = \psi((a_i)_{i \geq 1})$, and

$$\begin{aligned} \psi : \{0, 1, 2\}^{\mathbb{N}} &\longrightarrow \{0, 1, 2\}^{\mathbb{N}} \\ a_1 a_2 \dots &\longmapsto b_1 b_2 \dots \end{aligned}$$

is defined as follows: Put $b_1 = a_1$.

For $k \geq 2$, we define b_k as follows:

$$(29) \quad \text{If } a_k = 1 \text{ then } b_k = 1;$$

$$(30) \quad \text{If } a_k \neq 1 \text{ and } a_{k-1} = 2 \text{ then } b_k = a_k;$$

$$(31) \quad \text{If } a_k \neq 1 \text{ and } a_{k-1} = 0 \text{ then } b_k = 2 - a_k.$$

If $a_k \neq 1$ and $a_{k-1} = 1$, let $r = \min\{1 \leq i \leq k-1, a_i = a_{i+1} = \dots = a_{k-1} = 1\}$:

If $(r > 1 \text{ and } a_{r-1} = 2)$ or $r = 1$ then

$$(32) \quad \begin{cases} b_k = a_k, & \text{if } (k-r) \text{ is even} \\ b_k = 2 - a_k, & \text{if } (k-r) \text{ is odd} \end{cases}$$

where $(k-r)$ is the number of digits 1 after a number 0 or 2.

If $r > 1$ and $a_{r-1} = 0$ then

$$(33) \quad \begin{cases} b_k = a_k, & \text{if } (k-r) \text{ is odd} \\ b_k = 2 - a_k, & \text{se } (k-r) \text{ is even} \end{cases}$$

Theorem 6.6. : *The application $f : [0, 1] \longrightarrow \mathcal{R}_{\alpha-1}$ is well defined, bijective, continuous and $f(0) = -\alpha - \alpha^{-1}$, $f(1) = -1$.*

For the proof, we need the following classical Lemma.

Lemma 6.7. *Let t, t' be elements in $[0, 1]$, $t = \sum_{i=1}^{\infty} a_i 3^{-i}$, $t' = \sum_{i=1}^{\infty} c_i 3^{-i}$ with $a_i, c_i \in \{0, 1, 2\}$ such that $a_i = c_i$ for $i < k$ and $a_k < c_k$ for some $k \in \mathbb{N}^*$. Then*

- (1) *If $|t - t'| < 3^{-N}$, $N > k$ then $c_k = a_k + 1$ and $c_i = 0$, $a_i = 2$, $k+1 \leq i \leq N$.*
- (2) *If $t = t'$ then $c_k = a_k + 1$ and $c_i = 0$, $a_i = 2 \forall i \geq k+1$.*

Proof of Theorem 6.6: Let $t, t' \in [0, 1]$ such that $t = \sum_{i=1}^{\infty} a_i 3^{-i}$, $t' = \sum_{i=1}^{\infty} a'_i 3^{-i}$ with $a_i, a'_i \in \{0, 1, 2\}$, $a_i = a'_i$ for $i < k$, $a_k < a'_k$ for some integer $k \in \mathbb{N}$.

Assume that $f(t) = \lim_{n \rightarrow \infty} g_{b_1} \circ \dots \circ g_{b_n}(x_0)$, $f(t') = \lim_{n \rightarrow \infty} g_{b'_1} \circ \dots \circ g_{b'_n}(x_0)$ where $(b_i) = \psi(a_i)$ and $(b'_i) = \psi(a'_i)$.

f is well defined: Suppose that $t = t'$ and $(a_i)_{i \geq 1} \leq_{lex} (a'_i)_{i \geq 1}$. There exists an integer $k \in \mathbb{N}$ such that

$$t = a_1 \dots a_{k-1} c \bar{2} \quad \text{and} \quad t' = a_1 \dots a_{k-1} (c+1) \bar{0}$$

where $c = 0$ or 1 , $\bar{0} = 00000\dots$ and $\bar{2} = 2222\dots$

If $t = a_1 \dots a_{k-1} 0 \bar{2}$ and $t' = a_1 \dots a_{k-1} 1 \bar{0}$ then $\psi(t) = b_1 \dots b_{k-1} 20 \bar{2}$, $\psi(t') = b_1 \dots b_{k-1} 10 \bar{2}$ if $a_{k-1} = 0$. If $a_{k-1} = 1$, then $\psi(t) = b_1 \dots b_{k-1} x 0 \bar{2}$, $\psi(t') = b_1 \dots b_{k-1} 1 x \bar{2}$, where $x = 0$ or 2 . If $a_{k-1} = 2$, then $\psi(t) = b_1 \dots b_{k-1} 00 \bar{2}$, $\psi(t') = b_1 \dots b_{k-1} 1 \bar{2}$. By Lemma 6.5 and the fact that $h_2 \circ h_0(-1) = h_1 \circ h_0(-1)$, $h_x \circ h_0(-1) = h_1 \circ h_x(-1)$ for $x = 0$ or 2 and $h_0^2(-1) = h_1(-1)$ we deduce that $\psi(t) = \psi(t')$.

If $t = a_1 \dots a_{k-1} 1 \bar{2}$ and $t' = a_1 \dots a_{k-1} 2 \bar{0}$ then $\psi(t) = b_1 \dots b_{k-1} 1 \bar{2}$, $\psi(t') = b_1 \dots b_{k-1} 00 \bar{2}$ if $a_{k-1} = 0$, $\psi(t) = b_1 \dots b_{k-1} 1 x \bar{2}$, $\psi(t') = b_1 \dots b_{k-1} x 0 \bar{2}$, where $x = 0$ or 2 if $a_{k-1} = 0$ and $\psi(t) = b_1 \dots b_{k-1} 1 \bar{2}$, $\psi(t') = b_1 \dots b_{k-1} 20 \bar{2}$ if $a_{k-1} = 2$. As before, we deduce that $\psi(t) = \psi(t')$.

f is injective: We know that $a_i = a'_i$, $1 \leq i \leq k-1$. Then $f(t) = f(t') \iff h_{b_k}(z) = h_{b'_k}(z') \iff (b_k = 0, z = -\alpha - \alpha^{-1}, b'_k = 1, z' = -1)$ or $(b_k = 1, b'_k = 2, z = z' = -\alpha - \alpha^{-1})$. Hence, we have to consider the following cases:

- **Case 1** $b_k = 0$, $b'_k = 1$, $z = -\alpha - \alpha^{-1}$, $z' = -1$.

According to Proposition 6.1, we have $(b_i) = b_1 b_2 \dots b_{k-1} 00 \bar{2}$ and $(b'_i) = b_1 b_2 \dots b_{k-1} 1 \bar{2}$.

If $b'_k = 1$ then, by (29), $a'_k = 1$. As $b_k = 0$ we have to consider the following sub cases:

- **Case 1.1:** $a_{k-1} = 2$.

By (30), $a_k = 0$. We have $b_i = 2$ for all $i \geq k+1$. Hence by (30), $a_i = 2$, $\forall i \geq k+1$.

We also have $b'_j = 2$ for all $j \geq k+1$. By (31), $a'_j = 0$, $\forall j \geq k+1$. Hence,

$(a_i)_{i \geq 1} = a_1 a_2 \dots a_{k-2} 20 \bar{2}$ and $(a'_i)_{i \geq 1} = a_1 a_2 \dots a_{k-2} 21 \bar{0}$. Hence $t = t'$.

- **Case 1.2:** $a_{k-1} = 0$ and $b_k = 0$.

We have $b_{k+1} = 0$ and by (31) $a_k = 2$, $b_k = 0$ and $a_{k+1} = 2$. As $b_i = 2$, $\forall i \geq k+2$ then by (31) we have $a_i = 0$ for all $i \geq k+2$. Since $b'_j = 2$, $\forall j \geq k+1$ then by (30) we have

$a'_j = 2$, $\forall j \geq k+1$. Hence, $(a_i)_{i \geq 1} = a_1 a_2 \dots a_{k-2} 02 \bar{0}$ and $(a'_i)_{i \geq 1} = a_1 a_2 \dots a_{k-2} 01 \bar{2}$. Thus $t = t'$.

- **Case 1.3:** $a_{r-1} = 0$, $a_r = \dots = a_{k-1} = 1$ and $(k-r)$ is even.

If $b_k = 0$ then by (33) $a_k = 2$. Using (31) and (30) we have $(a_i)_{i \geq 1} = a_1 \dots a_{r-2} 011 \dots 12 \bar{0}$ and $(a'_i)_{i \geq 1} = a_1 \dots a_{r-2} 011 \dots 11 \bar{2}$. Hence $t = t'$.

Using the same arguments we prove the following cases: $a_{r-1} = 0, a_r = \dots = a_{k-1} = 1$ and $(k-r)$ odd, $a_{r-1} = 2, a_r = \dots = a_{k-1} = 1, (k-r)$ even, $a_{r-1} = 2, a_r = \dots = a_{k-1} = 1$ ($k-r$) odd, $a_1 = a_2 = \dots = a_{k-1} = 1$ and $(k-1)$ even and $a_1 = a_2 = \dots = a_{k-1} = 1$ and $(k-1)$ odd.

- **Case 2** $b_k = 1, b'_k = 2, z = z' = -\alpha - \alpha^{-1}$.

According to Proposition 6.1 we have $(b_i) = b_1 b_2 \dots b_{k-1} 10\bar{2}$ and $(b'_i) = b_1 b_2 \dots b_{k-1} 20\bar{2}$.

As $b_k = 1$ then $a_k = 1$. As $b'_{k+1} = 0$ using the same previous ideas we have to consider the following cases:

- **Case 2.1:** $a_{k-1} = 2$.

We have $(a_i)_{i \geq 1} = a_1 a_2 \dots a_{k-2} 21\bar{2}$ and $(a'_i)_{i \geq 1} = a_1 a_2 \dots a_{k-2} 22\bar{0}$. Then 6.7, $t = t'$.

- **Case 2.2:** $a_{k-1} = 0$.

We have $(a_i)_{i \geq 1} = a_1 a_2 \dots a_{k-2} 01\bar{0}$ and $(a'_i)_{i \geq 1} = a_1 a_2 \dots a_{k-2} 00\bar{2}$. Then $t = t'$.

- **Case 2.3:** $a_{r-1} = 0, a_r = \dots = a_{k-1} = 1$ and $(k-r)$ even.

Here we have $(a_i)_{i \geq 1} = a_1 \dots a_{r-2} 011 \dots 11\bar{0}$ and $(a'_i)_{i \geq 1} = a_1 \dots a_{r-2} 011 \dots 10\bar{2}$. Then $t = t'$.

Using the same arguments we prove the following cases: $a_{r-1} = 0, a_r = \dots = a_{k-1} = 1, (k-r)$ odd, $a_{r-1} = 2, a_r = \dots = a_{k-1} = 1, (k-r)$ even, $a_{r-1} = 2, a_r = \dots = a_{k-1} = 1, (k-r)$ odd, $a_1 = a_2 = \dots = a_{k-1} = 1, (k-1)$ even and $a_1 = a_2 = \dots = a_{k-1} = 1, (k-1)$ odd.

f is continuous: Let $t = \sum_{i=1}^{\infty} a_i 3^{-i}, t' = \sum_{i=1}^{\infty} a'_i 3^{-i}$ and suppose that $0 < |t - t'| < 3^{-N}, N \in \mathbb{N}, N > k$. By Lemma 6.7, $a'_k = a_k + 1, a'_i = 0$ and $a_i = 2$ for all i satisfying $k+1 \leq i \leq N$. We have to consider the following cases:

- **Case 1.1:** If $a_{k-1} = 0, a_k = 0, a'_k = 1$ then we can write

$$t = (a_1 \dots a_{k-2} 00\bar{2}), t' = (a_1 \dots a_{k-2} 01\bar{0})$$

and since $|h_i(z) - h_i(w)| \leq |\alpha|^2 |z - w|$ then

$$|f(t) - f(t')| = |f(t) - f(t')| = |h_2(z_1) - h_1(z'_1)| \cdot |\alpha|^{2(k-1)}.$$

Where $z_1 = h_0 h_2^{N-k-1}(y_1)$ and $z'_1 = h_0 h_2^{N-k-1}(y'_1), y_1, y'_1 \in \mathbb{C}$

$$|f(t) - f(t')| = |h_2(z_1) - h_1(z'_1)| \cdot |\alpha|^{2(k-1)}$$

As $h_2(-\alpha - \alpha^{-1}) = h_1(-\alpha - \alpha^{-1}) = -(1 + \alpha^2 + \alpha^4)$ then

$$\begin{aligned} |f(t) - f(t')| &\leq |h_2(z_1) - h_2(-\alpha - \alpha^{-1})| + |h_1(-\alpha - \alpha^{-1}) - h_1(z'_1)| \times |\alpha|^{2(k-1)} \\ &\leq (1 + |\alpha|) |\alpha|^{2k+1} \text{diam}(\mathcal{R}_{\alpha-1}), \end{aligned}$$

where $\text{diam}(\mathcal{R}_{\alpha-1})$ is the diameter of $\mathcal{R}_{\alpha-1}$.

- **Case 1.2:** If $a_{k-1} = 2, a_k = 0, a'_k = 1$ then we can write

$$t = (a_1 \dots a_{k-2} 20\bar{2}), t' = (a_1 \dots a_{k-2} 21\bar{0}).$$

Then

$$|f(t) - f(t')| = |h_2(z_2) - h_1(z'_2)| \cdot |\alpha|^{2(k-1)},$$

where $z_2 = h_0 h_2^{N-k-1}(y_2)$ and $z'_2 = h_0 h_2^{N-k-1}(y'_2), y_2, y'_2 \in \mathbb{C}$. Hence

$$|f(t) - f(t')| \leq (1 + |\alpha|) |\alpha|^{2k+1} \text{diam}(\mathcal{R}_{\alpha-1}).$$

- **Case 1.3:** If $a_{i-1} = 0, a_i = \dots = a_{k-1} = 1, a_k = 0, a'_k = 1$ and $(k-i-1)$ is even. This case can be done by the same as before and is left to the reader.

□

7. HAUSDORFF DIMENSION OF $\partial(\mathcal{R}_2)$

Since $\partial(\mathcal{R}_2)$ is the union of 4 curves that are images of $\mathcal{R}_{\alpha-1}$ by a affine application, we have that $\dim_H \mathcal{R}_{\alpha-1} = \dim_H \partial \mathcal{R}$. By Proposition 6.1, the set $\mathcal{R}_{\alpha-1} = \cup_{i=0}^2 h_i(\mathcal{R}_{\alpha-1})$ is invariant by the affine maps h_i . An upper bound of Hausdorff dimension of this class of sets is given by Theorem

Theorem 7.1. [12] *Let A a set of \mathbb{C} such that $A = \cup_{i=0}^n \varphi_i(A)$ is compact and invariant for affine applications φ_i with coefficients r_i (i.e. , $\forall x, y \in \mathbb{C}, |\varphi_i(x) - \varphi_i(y)| = r_i |x - y|$), then $\dim_H(A) \leq s$, where s is the unique real number that verifies $\sum_{i=0}^n r_i^s = 1$.*

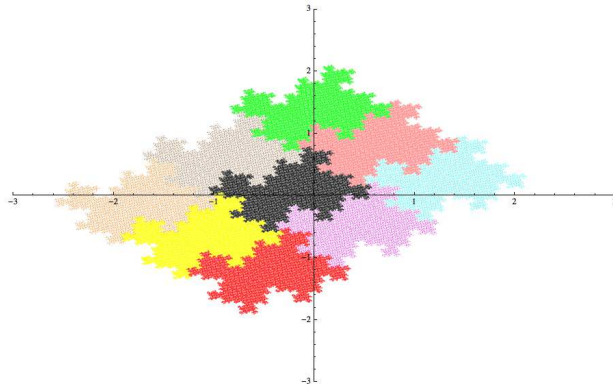
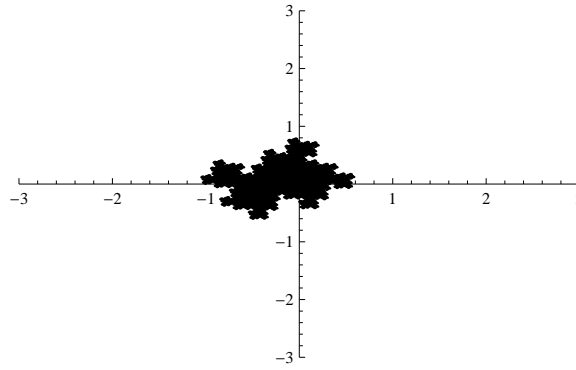
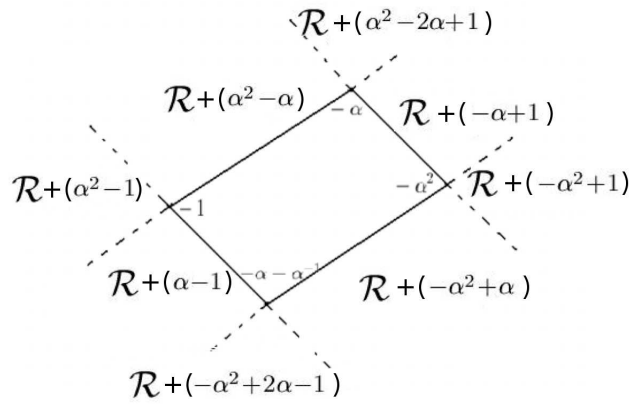
Remark 7.2. *When the $\varphi_i(A)$ intercept in points is known that $\dim_H(A) = s$ (see [12]).*

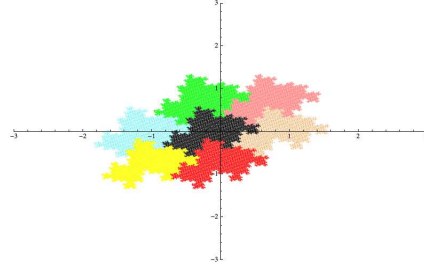
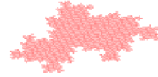
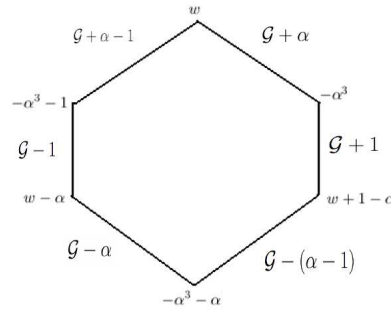
By Proposition 6.1, we deduce that $\dim_H(\partial(\mathcal{R}_2)) = s$, where s verifies

$$|\alpha|^{2s} + |\alpha|^{3s} + |\alpha|^{4s} = 1.$$

Therefore $\dim_H(\partial(\mathcal{R}_2)) = \frac{\log \rho}{\log |\alpha|} \approx 1.359337357$, where ρ is the root is the maximum real root of the polynomial $X^4 + X^3 + X^2 - 1 = 0$.

Remark 7.3. *Using the automaton \mathcal{A} , we can also parametrize \mathcal{G}_2 and prove that it is homeomorphic to a topological disk. We can also show that $\dim_H(\partial(\mathcal{G}_2)) = \dim_H(\partial(\mathcal{R}_2))$. All these results can be extended to all \mathcal{R}_a and \mathcal{G}_a , $a \geq 3$.*


FIGURE 1. Fractal \mathcal{R}_a

FIGURE 2. A tile of Fractal \mathcal{R}_a

FIGURE 3. Boundary's Fractal \mathcal{R}_a

FIGURE 4. Fractal \mathcal{G}_a FIGURE 5. A tile of Fractal \mathcal{G}_a FIGURE 6. Boundary's Fractal \mathcal{G}_a

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